Seismology and Seismic Hazard

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Introduction

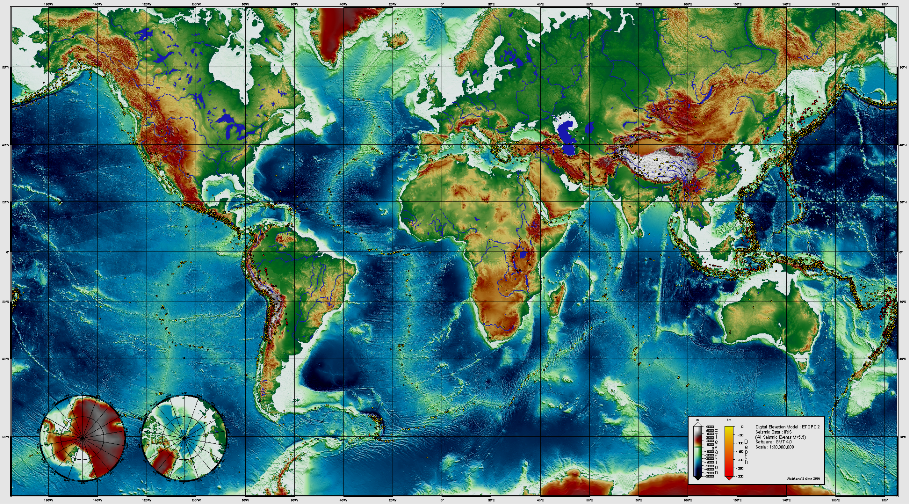
Seismology

- Comprises all about earthquakes and the propagation of seismic waves in the Earth.
- One of the main fields of solid-earth geophysics.
- Has provided the majority of our knowledge on Earth's interior.

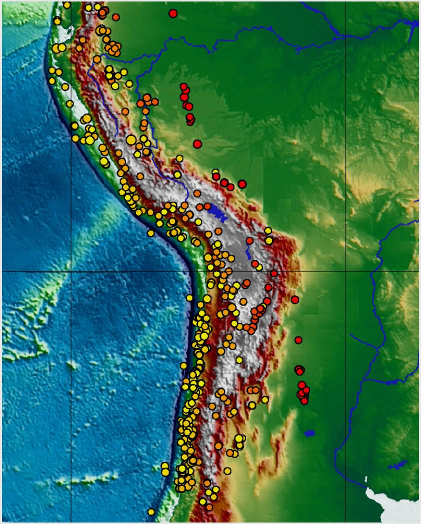
Seismics

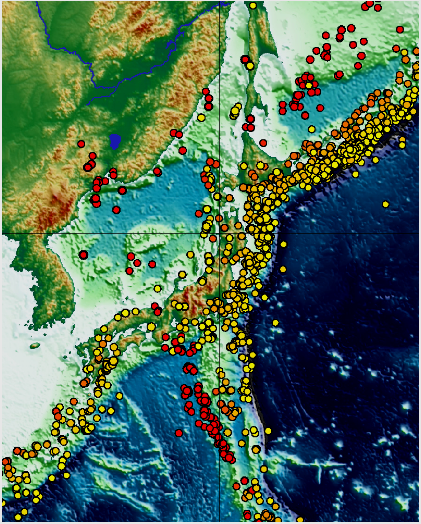
- Exploration of the deep and shallow subsurface with the help of artificial seismic waves.
- The perhaps most important field of applied geophysics.

Worldwide Distribution of Earthquakes



Worldwide Distribution of Earthquakes





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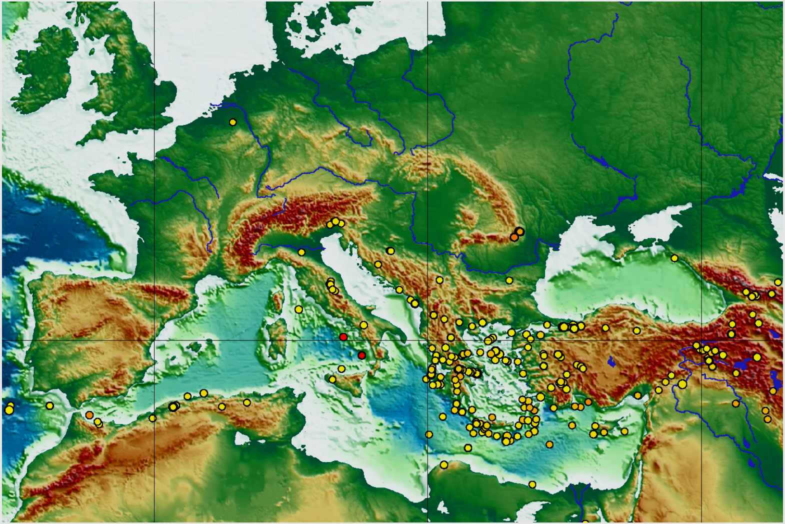
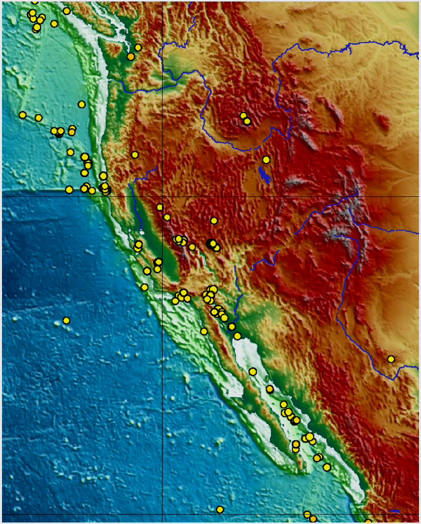
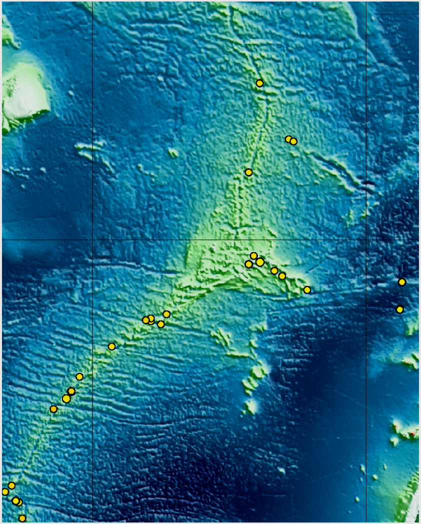
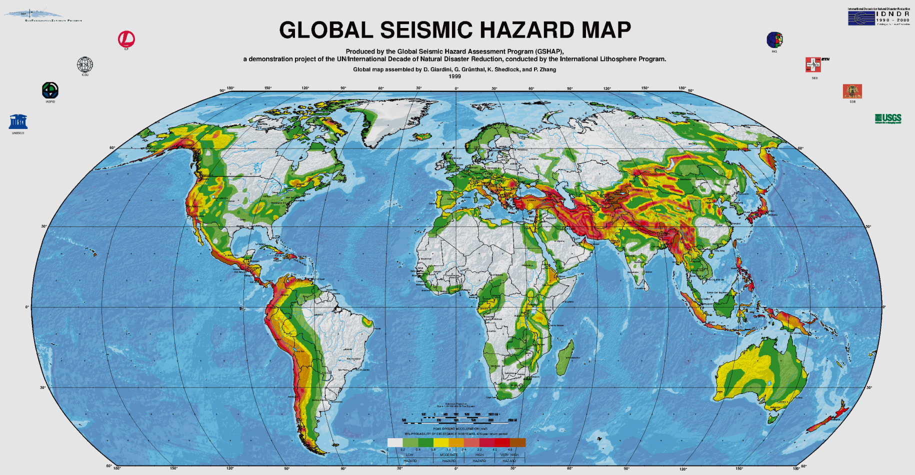


Worldwide Distribution of Earthquakes



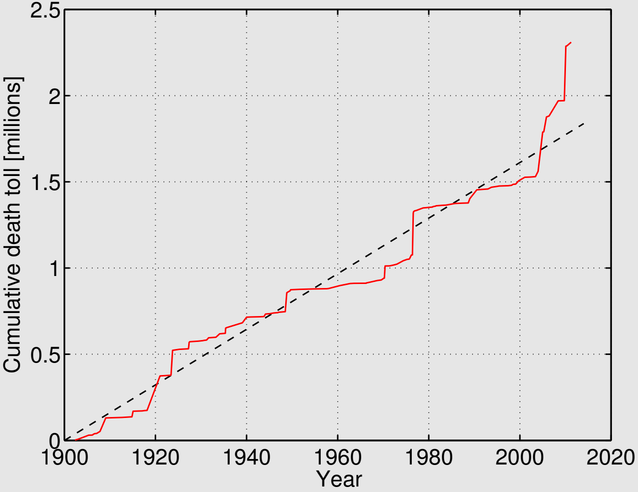


Earthquake Hazard



Source: Global Seismic Hazard Assessment Program

Earthquakes with 1000 or more Deaths 1900–2014



begineqnarray-1ex] Data: USGS Earthquake Hazards Program

The First “Seismometer” (132 a.D.)

History of Seismology

1660	basic law of elasticity	R. Hooke
1821–22	differential equations of elasticity	C. Navier A. L. Cauchy
1830	theory of two fundamental types of elastic waves (P- and S-wave)	S. D. Poisson
1875	First "serious" seismometer	F. Gecchi
1887	theory of the first type of surface waves	J. W. Strutt (3. Lord Rayleigh)
1889	first recording of a distant earthquake	
1892	first compact seismometer, used at about 40 stations	J. Milne
1894	statistics of aftershocks	F. Omori
1903	12 degree scale for the intensity of earthquakes based on the damage	G. Mercalli

History of Seismology

1906–1913	detection of the liquid core of the earth and determination of its size	R. D. Oldham, B. Gutenberg
1909	detection of the crust-mantle discontinuity	A. Mohorovičić
1911	theory of a second type of surface waves	A. E. H. Love
1935	local magnitude as an “objective” measure of earthquake intensity	C. F. Richter
1936	detection of the inner, solid core	I. Lehmann
1954	frequency-magnitude relation of earthquakes	B. Gutenberg, C. F. Richter
1975	first successful short-term prediction of a strong earthquake	
1977	moment magnitude as a measure of earthquake source strength	H. Kanamori

The Navier-Cauchy Equations in Seismology

- Small, but spatially and temporally variable displacement $\vec{u}(\vec{x}, t)$
- Neglect gravity
- Sign convention as in mathematics, physics, and engineering
- Elastic deformation

$$\Downarrow$$

$$\rho \frac{\partial^2}{\partial t^2} \vec{u} = \begin{pmatrix} \frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3} \\ \frac{\partial \sigma_{21}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{23}}{\partial x_3} \\ \frac{\partial \sigma_{31}}{\partial x_1} + \frac{\partial \sigma_{32}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} \end{pmatrix} = \mathrm{div}(\boldsymbol{\sigma}) \tag{1}$$

with the stress tensor

$$\boldsymbol{\sigma} = \lambda \, \epsilon_v \, \mathbf{1} + 2\mu \, \boldsymbol{\epsilon}, \tag{2}$$

The Navier-Cauchy Equations in Seismology

the strain tensor $\boldsymbol{\epsilon}$ consisting of the components

$$\epsilon_{ij} \;=\; \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \tag{3}$$

the volumetric strain

$$\epsilon_v \;=\; \epsilon_{11} + \epsilon_{22} + \epsilon_{33}, \tag{4}$$

the density ρ and the Lamé parameters of the medium λ and μ

The Navier-Cauchy Equations in 1D

Displacement $u(x,t)$ instead of $\vec{u}(\vec{x},t)$.

$$\Downarrow$$

$$\rho \frac{\partial^2}{\partial t^2} u = \frac{\partial}{\partial x} \sigma \tag{5}$$

with

$$\sigma = (\lambda + 2\mu)\,\epsilon = (\lambda + 2\mu)\,\frac{\partial}{\partial x} u \tag{6}$$

$$\Downarrow$$

$$\rho \frac{\partial^2}{\partial t^2} u(x,t) = \frac{\partial}{\partial x}\left((\lambda + 2\mu)\,\frac{\partial}{\partial x} u(x,t)\right) \quad \text{(1D wave equation)} \tag{7}$$

Solution of the 1D Wave Equation

If λ and μ are constant:

$$\rho \frac{\partial^2}{\partial t^2} u(x,t) = (\lambda + 2\mu) \frac{\partial^2}{\partial x^2} u(x,t) \tag{8}$$

Solution:

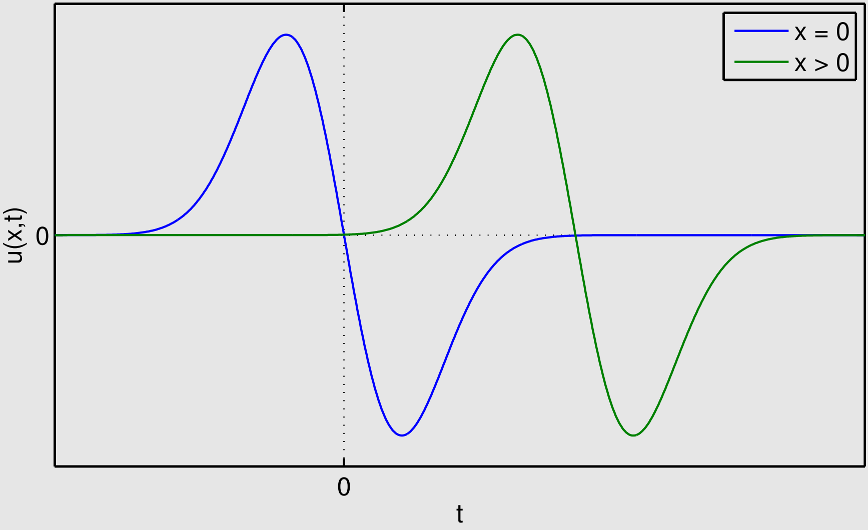
$$u(x,t) = f(t \pm sx) \tag{9}$$

where

$$\begin{aligned} f &= \text{arbitrary function describing the shape of the wave} \\ s &= \sqrt{\frac{\rho}{\lambda + 2\mu}} = \text{slowness} \end{aligned}$$

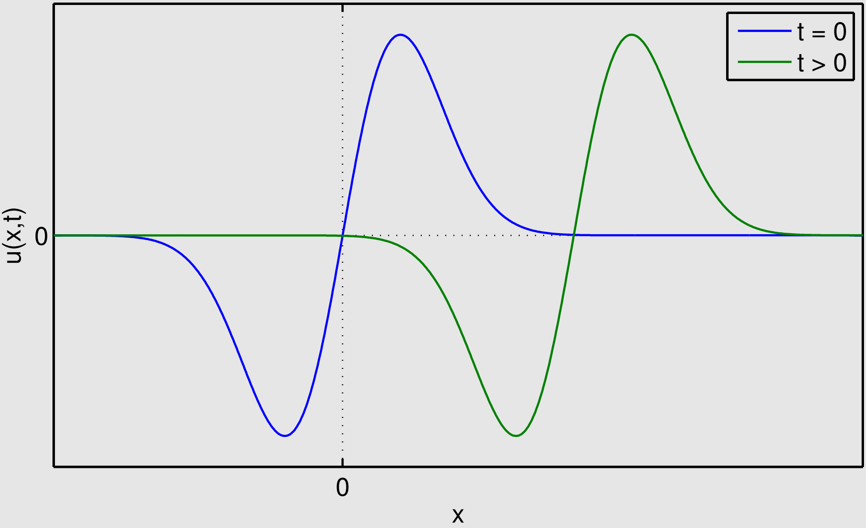
The wave moves in positive or negative x direction with a velocity $v = \frac{1}{s}$.

Example



One-Dimensional Wave Propagation

Example



The Retarded Time

$\tau = t \pm sx$ is called retarded time.

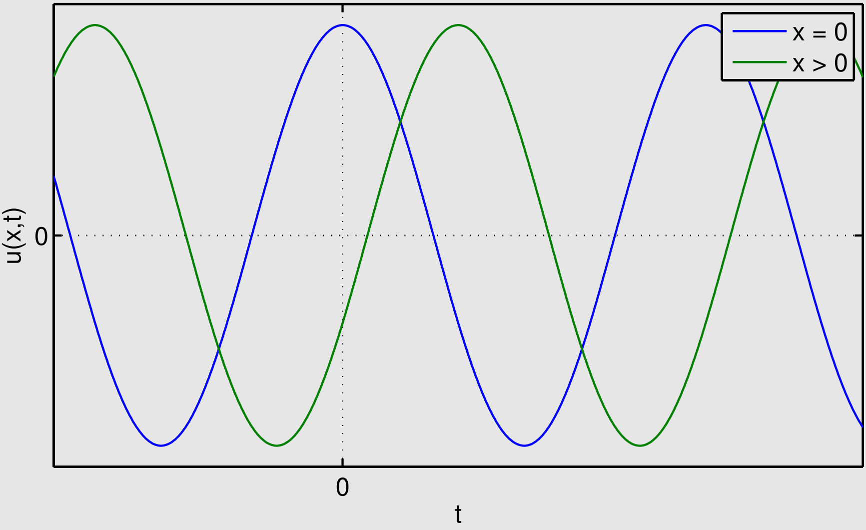
Meaning: At the position x and time t we observe what happened at the origin ($x = 0$, e. g. earthquake focus) at the retarded time $\tau = t \pm sx$.

The Shape of the Wave

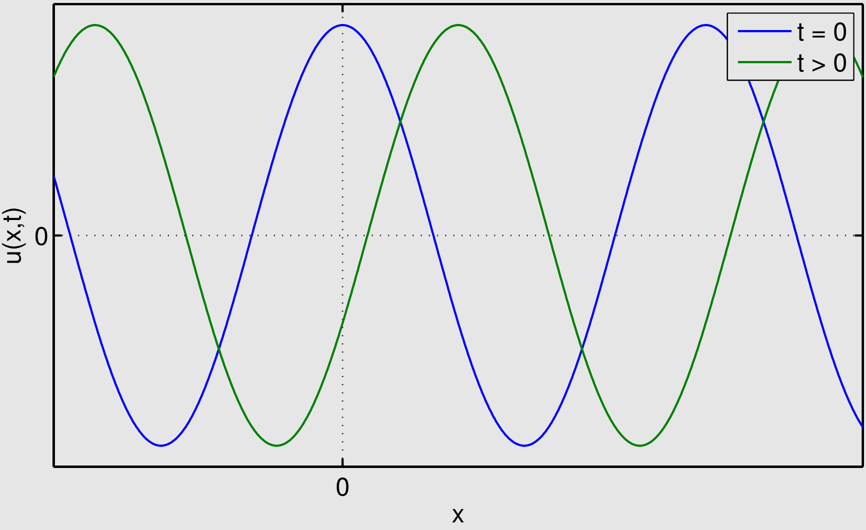
Examples for the function f:

- $f(\tau) = \begin{cases} 1 & \text{for } \tau \geq T \\ 0 & \text{else} \end{cases}$ describes a step-like shape (shock wave).
- $f(\tau) = \begin{cases} 1 & \text{for } |\tau| \leq \frac{T}{2} \\ 0 & \text{else} \end{cases}$ describes a boxcar-shaped wave.
- $f(\tau) = a\cos(\omega\tau)$, $f(\tau) = a\sin(\omega\tau)$ or $f(\tau) = a\,e^{i\omega\tau}$ describes a harmonic wave with an angular frequency ω and amplitude a.

Basic Terms



Basic Terms

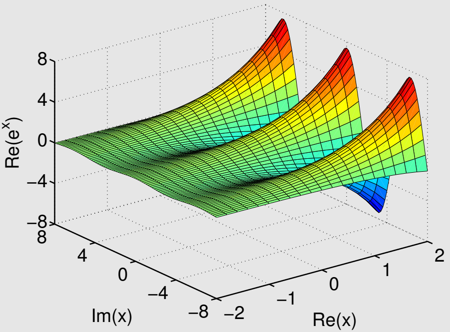


Basic Terms

- Time domain:
  Angular frequency: ω $[\frac{1}{\mathrm{s}}]$
  Frequency: $\nu = \frac{\omega}{2\pi}$ $[\frac{1}{\mathrm{s}}]$
  Period: $T = \frac{1}{\nu} = \frac{2\pi}{\omega}$ $[\mathrm{s}]$
- Spatial wave pattern:
  Wave number: $k = \omega s$ $[\frac{1}{\mathrm{m}}]$
  Wavelength: $L = \frac{2\pi}{k} = \frac{1}{\nu s}$ $[\mathrm{m}]$

The Complex Exponential Function vs. Sine and Cosine

With $e^{i\phi} = \cos\phi + i\sin\phi$, the complex exponential function combines the real exponential function with the sine and cosine functions.



The Complex Exponential Function vs. Sine and Cosine

$$\mathrm{Re}\left(e^{i\omega\tau}\right) = \cos(\omega\tau) \quad \text{and} \quad \mathrm{Im}\left(e^{i\omega\tau}\right) = \sin(\omega\tau) \tag{10}$$

$$\Downarrow$$

Real part and imaginary part of the complex solution can be considered as independent real solutions.

Derivatives of the complex solutions are simpler than those of the real solutuions:

$$\frac{\partial}{\partial\tau} e^{i\omega\tau} = i\omega e^{i\omega\tau} \tag{11}$$

while

$$\frac{\partial}{\partial\tau}\cos(\omega\tau) = -\omega\sin(\omega\tau) \quad \text{and} \quad \frac{\partial}{\partial\tau}\sin(\omega\tau) = \omega\cos(\omega\tau) \tag{12}$$

Seismic Waves in 3D

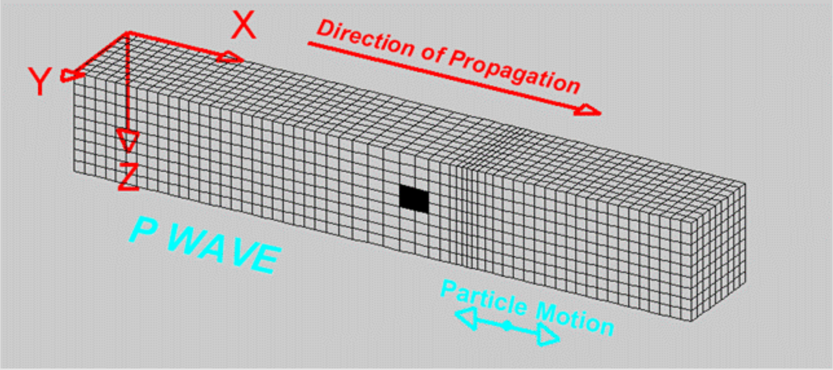
Complications towards the 1D case:

- Displacement $\vec{u}(\vec{x}, t)$ is a vector.
- Propagation in arbitrary direction in space instead of the positive or negative x axis only.

Fundamental Types of Body Waves

Two types of independent plane waves in an infinite, homogeneous elastic medium:

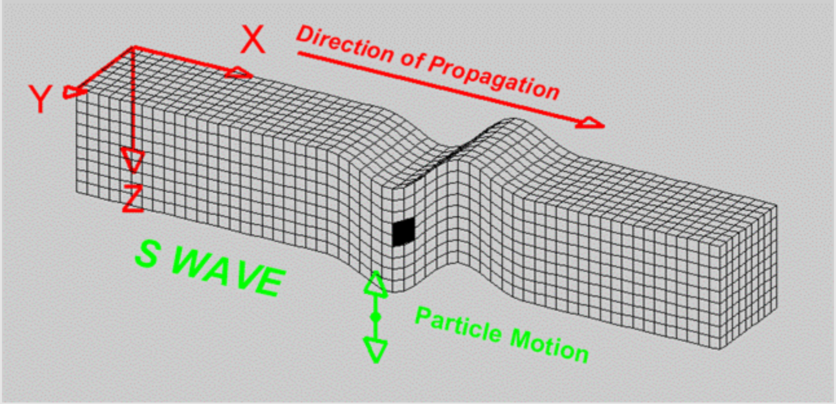
- Compressional wave (longitudinal wave, primary wave)



Source: L. Braile, Purdue University

Fundamental Types of Body Waves

- Shear wave (transverse wave, secondary wave)



Source: L. Braile, Purdue University

Plane Waves in Infinite, Homogeneous, and Isotropic Media

Plane wave: $\vec{u}(\vec{x}, t)$ is constant on parallel planes.

Mathematical description:

$$\vec{u}(\vec{x}, t) = f(t - \vec{s} \cdot \vec{x})\, \vec{a} \tag{13}$$

or for a harmonic wave:

$$\vec{u}(\vec{x}, t) = e^{i\omega(t - \vec{s} \cdot \vec{x})}\, \vec{a} \tag{14}$$

where

$$\begin{aligned} \vec{s} &= \text{slowness vector} \\ \vec{a} &= \text{amplitude vector (constant)} \end{aligned}$$

The wave moves in direction of $\vec{s}$ with a velocity $v = \frac{1}{|\vec{s}|}$.

Plane Waves in Infinite, Homogeneous, and Isotropic Media

Simplest version: propagation in x_1 direction, $\vec{s} = \begin{pmatrix} s \\ 0 \\ 0 \end{pmatrix}$

$\Downarrow$

$$\vec{u}(\vec{x}, t) = f(t - sx_1)\,\vec{a} \tag{15}$$

$\Downarrow$

$$\epsilon = -s\,f'(t - sx_1) \begin{pmatrix} a_1 & \frac{1}{2}a_2 & \frac{1}{2}a_3 \\ \frac{1}{2}a_2 & 0 & 0 \\ \frac{1}{2}a_3 & 0 & 0 \end{pmatrix} \tag{16}$$

$\Downarrow$

$$\boldsymbol{\sigma} = -s\,f'(t - sx_1) \begin{pmatrix} (\lambda + 2\mu)a_1 & \mu a_2 & \mu a_3 \\ \mu a_2 & \lambda a_1 & 0 \\ \mu a_3 & 0 & \lambda a_1 \end{pmatrix} \tag{17}$$

Plane Waves in Infinite, Homogeneous, and Isotropic Media

If λ and μ are constant:

$$\operatorname{div}(\boldsymbol{\sigma}) = s^2 f''(t - sx_1) \begin{pmatrix} (\lambda + 2\mu)a_1 \\ \mu a_2 \\ \mu a_3 \end{pmatrix} \tag{18}$$

Insert into the Navier-Cauchy equations:

$$\rho \frac{\partial^2}{\partial t^2} \vec{u} = \rho f''(t - sx_1)\, \vec{a} \tag{19}$$

$$= \operatorname{div}(\boldsymbol{\sigma}) = s^2 f''(t - sx_1) \begin{pmatrix} (\lambda + 2\mu)a_1 \\ \mu a_2 \\ \mu a_3 \end{pmatrix} \tag{20}$$

Can only be satisfied if $a_2 = a_3 = 0$ and $s = \sqrt{\frac{\rho}{\lambda + 2\mu}}$ (longitudinal polarization) or $a_1 = 0$ and $s = \sqrt{\frac{\rho}{\mu}}$ (transverse polarization).

Plane Waves in Infinite, Homogeneous, and Isotropic Media

General case considered in assignment 2: Navier-Cauchy equations can be satisfied only if either $\vec{a}$ is parallel (or opposite) to $\vec{s}$ or normal to $\vec{s}$.

Transverse wave: $\vec{a}$ is normal to $\vec{s}$ ($\vec{a} \cdot \vec{s} = 0$)

$$|\vec{s}|^2 = \frac{\rho}{\mu}, \quad v_s = \frac{1}{|\vec{s}|} = \sqrt{\frac{\mu}{\rho}} \tag{21}$$

Longitudinal wave: $\vec{a}$ is parallel or opposite to $\vec{s}$

$$|\vec{s}|^2 = \frac{\rho}{\lambda + 2\mu}, \quad v_p = \frac{1}{|\vec{s}|} = \sqrt{\frac{\lambda + 2\mu}{\rho}} \tag{22}$$

Seismic Body Waves

Comparison with Sound Waves in Liquids and Gases

The longitudinal wave is similar to sound waves in liquids and gases, while the transverse wave has no counterpart in liquids and gases.

Seismic Velocities

Medium	Longitudinal wave $[\frac{km}{s}]$	Transverse wave $[\frac{km}{s}]$
air	0.34	–
water	1.45	–
wood	about 3	about 1.8
Earth*	5.8–13.7	3.4–7.2

*Parametric Earth Models (PEM), not valid for the shallow subsurface

Seismic Velocities

$$\frac{v_p}{v_s} = \sqrt{\frac{\lambda + 2\mu}{\mu}} \geq \sqrt{\frac{4}{3}} \approx 1.15 \tag{23}$$

$\Downarrow$

Velocity v_p of the longitudinal wave is always higher than the velocity v_s of the transverse wave.

$\Downarrow$

Longitudinal wave always arrives prior to the transverse wave.

$\Downarrow$

longitudinal wave = primary wave (P-wave)
transverse wave = secondary wave (S-wave)

Typical v_p-v_s Ratios

- For solid rocks:

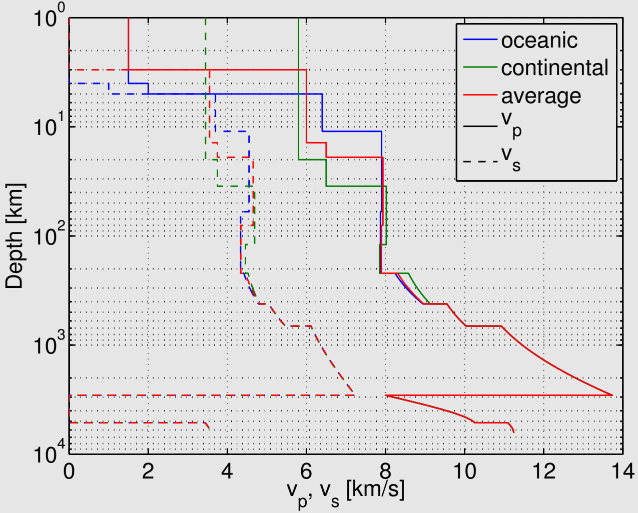
$$\frac{v_p}{v_s} = \sqrt{\frac{\lambda + 2\mu}{\mu}} \approx \sqrt{3} \approx 1.7 \tag{24}$$

  for $\lambda \approx \mu$.

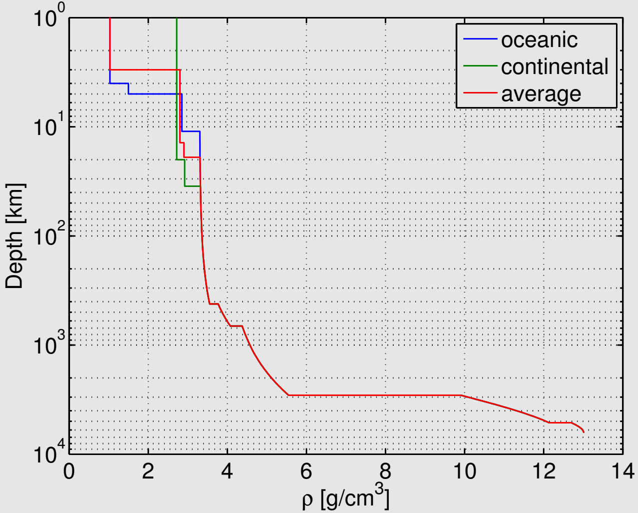
- For soil or unconsolidated rocks:

$$\frac{v_p}{v_s} \approx 2.5 \tag{25}$$

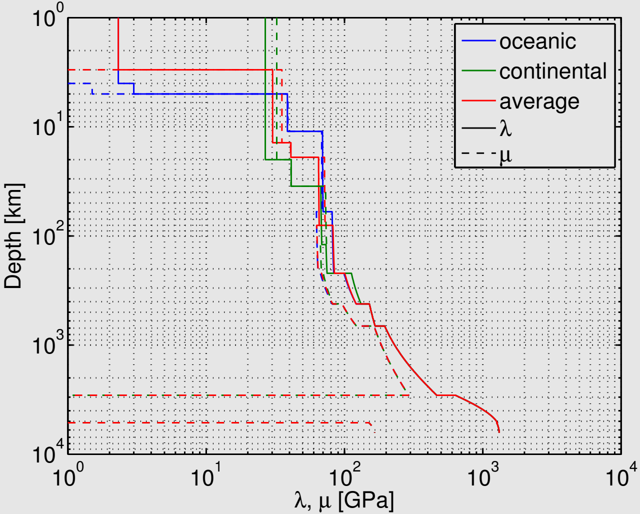
Seismic Velocities according to the Parametric Earth Models (PEM)



Density according to the Parametric Earth Models (PEM)



Lamé Parameters according to the Parametric Earth Models (PEM)



Typical P-wave Velocities in the Shallow Subsurface

Medium	v_p $[\frac{km}{s}]$
weathering zone	0.1–0.5
dry sand	0.3–0.6
water-saturated sand	1.3–1.8
sandstone	1.8–4
pit coal	1.6–1.9

Medium	v_p $[\frac{km}{s}]$
clay	1.2–2.8
claystone	2.2–4.2
limestone	3–6
halite	4.5–6.5
granite	5–6.5

Ray Optics in Seismology

Extensions towards the plane wave approach: Replace

$$\vec{u}(\vec{x},t) \;=\; f(t-\vec{s}\cdot\vec{x})\,\vec{a} \tag{26}$$

by

$$\vec{u}(\vec{x},t) \;=\; f(t-\psi(\vec{x}))\,\vec{a}(\vec{x}) \tag{27}$$

- Retarded time $\tau = t - \psi(\vec{x})$ instead of $\tau = t - \vec{s}\cdot\vec{x}$ with a general phase function $\psi(\vec{x})$
- Spatially variable amplitude vector $\vec{a}(\vec{x})$

$$\Downarrow$$

- Surfaces where $\psi(\vec{x})$ is constant define wave fronts (no longer planes).
- Wave propagates locally in direction of $\nabla\psi(\vec{x})$.

Ray Optics in Seismology

Compute $\boldsymbol{\sigma}(\vec{x}, t)$ and insert it into the Navier-Cauchy equations.

$$\Downarrow$$

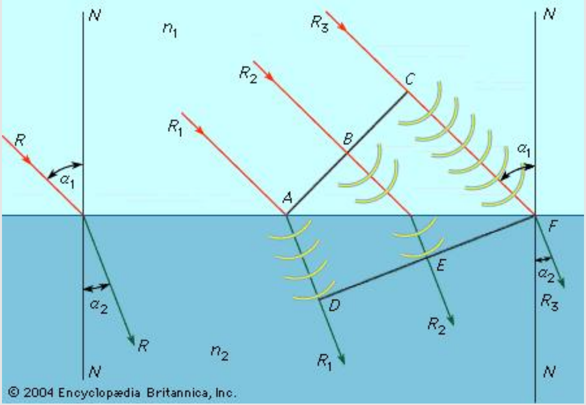
$\vec{a}(\vec{x})$ must be either parallel (or opposite) or normal to $\nabla\psi(\vec{x})$, and

$$|\nabla\psi(\vec{x})|^2 \;=\; \frac{\rho}{\lambda + 2\mu} \quad \text{or} \quad |\nabla\psi(\vec{x})|^2 \;=\; \frac{\rho}{\mu} \quad \text{(Eikonal equation).} \tag{28}$$

Change in amplitude in 1D (amplification of waves, e. g., by sediment layers) is considered in assignment 3.

Reflection and Refraction

Simplest case: two homogeneous, isotropic halfspaces with different properties (λ, μ, ρ) and plane waves in each of them.

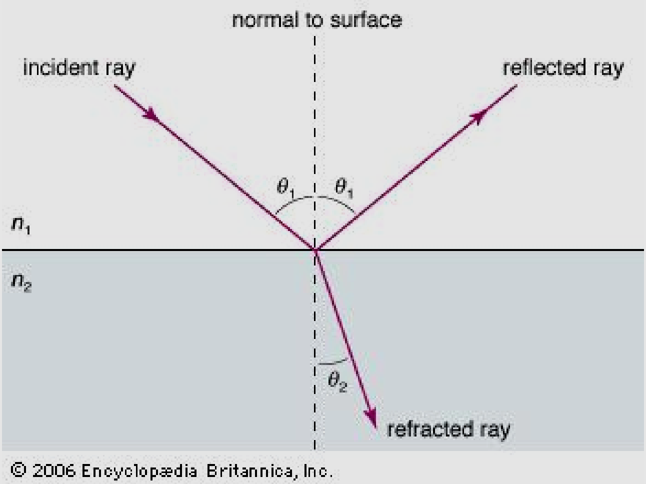


Source: Encyclopaedia Britannica

Snell's law:

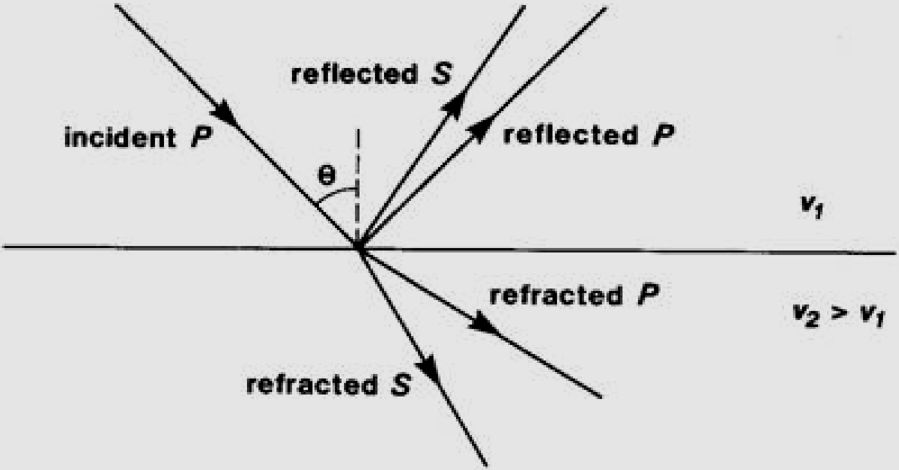
$$\frac{\sin \alpha_1}{\sin \alpha_2} = \frac{v_1}{v_2} \qquad (29)$$

Reflection and Refraction



Source: Encyclopaedia Britannica

Reflection and Refraction



Source: University College London

Mathematical Description of Reflection and Refraction

Two homogeneous halfspaces separated by the plane $x_3 = 0$, several harmonic plane waves in each halfspace.

Upper halfspace ($x_3 > 0$): incident wave (any type), reflected P-wave, reflected S-wave; parameters ρ_1, λ_1, and μ_1

Lower halfspace ($x_3 < 0$): transmitted (refracted) P-wave, transmitted (refracted) S-wave; parameters ρ_2, λ_2, and μ_2

Each wave is characterized by an angular frequency ω, a slowness vector $\vec{s}$ and an amplitude vector $\vec{a}$.

$$\vec{u}(\vec{x}, t) \;=\; e^{i\omega(t-\vec{s}\cdot\vec{x})}\,\vec{a} \tag{30}$$

Mathematical Description of Reflection and Refraction

Conditions at the interface:

(1) Displacement must be continuous:

$$\sum_{\text{upper halfspace}} \vec{u}(\vec{x},t)\big|_{x_3=0} = \sum_{\text{lower halfspace}} \vec{u}(\vec{x},t)\big|_{x_3=0} \tag{31}$$

$$\Downarrow$$

$$\sum_{\text{upper halfspace}} e^{i\omega(t-s_1x_1-s_2x_2)}\,\vec{a} = \sum_{\text{lower halfspace}} e^{i\omega(t-s_1x_1-s_2x_2)}\,\vec{a} \tag{32}$$

$$\Downarrow$$

ω, s_1, s_2 must be the same for all 5 involved waves, and

$$\sum_{\text{upper halfspace}} \vec{a} = \sum_{\text{lower halfspace}} \vec{a} \tag{33}$$

Mathematical Description of Reflection and Refraction

$$\begin{aligned} s_1, s_2 &= \text{horizontal components of } \vec{s} \text{ with regard to the interface} \\ s_3 &= \text{vertical component of } \vec{s} \text{ with regard to the interface} \end{aligned}$$

s_1, s_2 are the same for all involved waves.

$$\Downarrow$$

General form of Snell's law:

Horizontal slowness remains constant in reflection and refraction.

- Horizontal velocity is not constant!
- Conservation of horizontal slowness is the main reason why slowness is preferred to veloctiy in seismology.

Mathematical Description of Reflection and Refraction

Second condition at the interface:

(2) Stress vector acting on the interface, given by

$$\vec{\sigma}_{\text{int}} = \boldsymbol{\sigma} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = -\, i\omega\, e^{i\omega(t - s_1 x_1 - s_2 x_2)} \begin{pmatrix} \mu s_3 a_1 + \mu s_1 a_3 \\ \mu s_3 a_2 + \mu s_2 a_3 \\ \lambda\,(s_1 a_1 + s_2 a_2 + s_3 a_3) + 2\mu s_3 a_3 \end{pmatrix} \tag{34}$$

must be continuous,

$$\sum_{\text{upper halfspace}} \vec{\sigma}_{\text{int}}\big|_{x_3=0} = \sum_{\text{lower halfspace}} \vec{\sigma}_{\text{int}}\big|_{x_3=0} \tag{35}$$

The entire stress tensor $\boldsymbol{\sigma}$ is not necessarily constant!

(1) + (2) $\Rightarrow$ linear equation system for the amplitude vectors $\vec{a}$ of the five involved waves.

Conversion of Waves in Reflection and Refraction

Align the coordinate system in such a way that all waves propagate in the x_1-x_3 plane ($s_2 = 0$, possible because s_1 and s_2 are the same for all involved waves).

$\Downarrow$

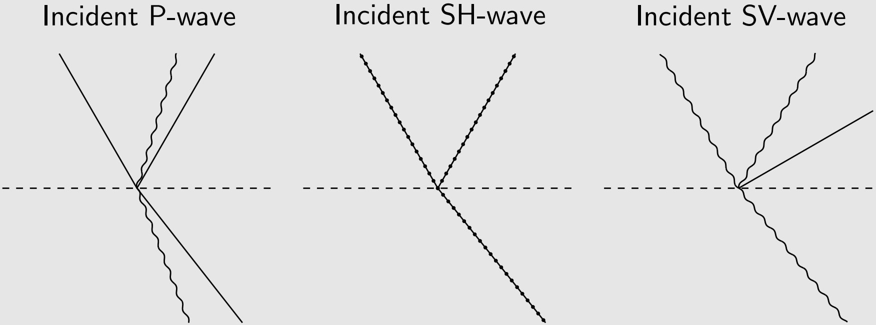
Vertically polarized S-wave (SV-wave):

- $a_2 = 0$ $\rightarrow$ particle displacement in the x_1-x_3 plane (and normal to wave propagation)
- Converted to (and from) P and SV waves in reflection and refraction

Horizontally polarized S-wave (SH-wave):

- $a_1 = a_3 = 0$ $\rightarrow$ particle displacement in x_2 direction
- Independent of P and SV waves

Conversion of Waves in Reflection and Refraction

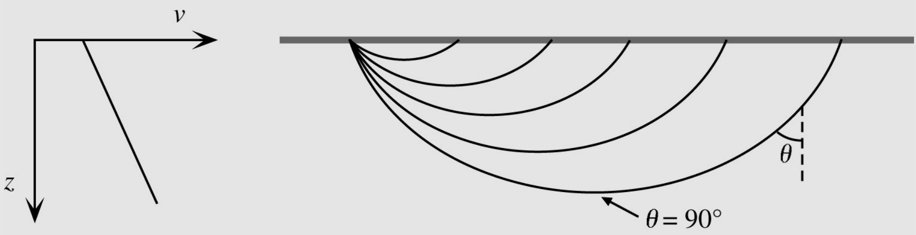


Wave Propagation in an Almost Homogeneous Medium

- No reflection and no conversion of P- and S-waves.
- Ray path can be computed from the condition that the horizontal slowness is constant.
- Alternatively: Approximate the continuous change in the velocity by many small discrete steps.
- Result: Continuous refraction towards regions of lower velocity.

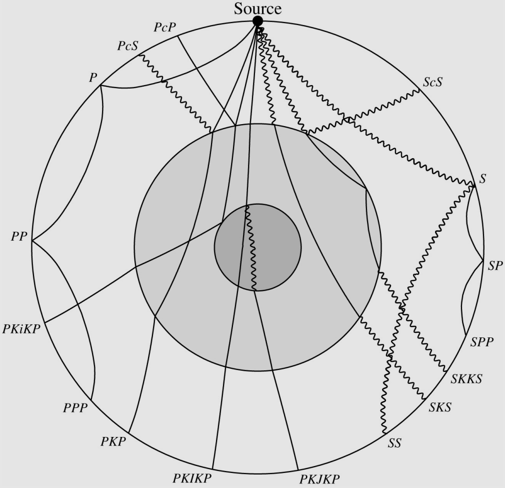
Wave Propagation in an Almost Homogeneous Medium

Example: velocity continuously increasing width depth



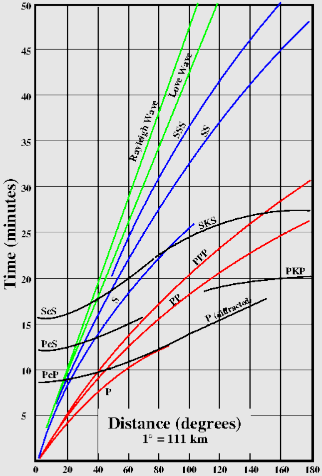
Source: Shearer, Introduction to Seismology

Global Wave Propagation in the Earth's Interior



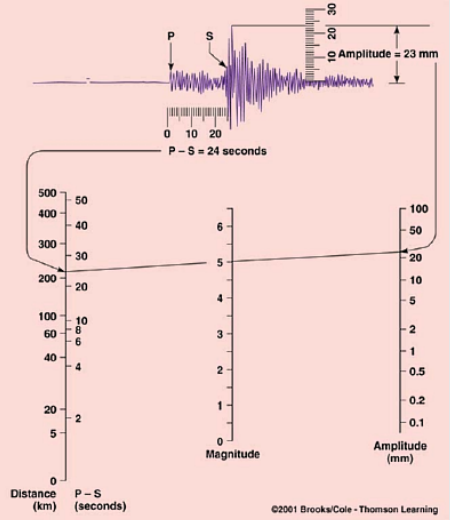
Source: Shearer, Introduction to Seismology

Travel Time Curves



Source: Southern Arizona Seismic Observatory

Localization of Earthquakes



The Waves Disappearing in Reflection and Refraction

For all waves involved in refection and refraction at a planar interface ($s_2 = 0$): s_1 given and

$$|\vec{s}|^2 \;=\; s_1^2 + s_3^2 \;=\; \begin{cases} \frac{1}{v_p^2} = \frac{\rho}{\lambda+2\mu} & \text{for P-waves} \\ \frac{1}{v_s^2} = \frac{\rho}{\mu} & \text{for S-waves} \end{cases} \tag{36}$$

given.

$$\Downarrow$$

$$s_3 \;=\; \pm\sqrt{|\vec{s}|^2 - s_1^2} \;=\; \pm\, i\sqrt{s_1^2 - |\vec{s}|^2} \quad \text{if} \quad s_1 > |\vec{s}| \tag{37}$$

Introduce the aspect ratio

$$S \;=\; \left|\frac{s_3}{s_1}\right| \;=\; \sqrt{1 - \frac{|\vec{s}|^2}{s_1^2}} \tag{38}$$

Harmonic Interface Waves

For a harmonic wave:

$$\vec{u}(\vec{x}, t) = e^{i\omega(t - \vec{s}\cdot\vec{x})}\,\vec{a} = e^{i\omega(t - s_1 x_1 \mp i s_1 S x_3)}\,\vec{a} \tag{39}$$

$$= e^{i\omega(t - s_1 x_1)}\,e^{\pm\omega s_1 S x_3}\,\vec{a} = e^{i\omega(t - s_1 x_1)}\,\vec{a}_{\mathrm{eff}} \tag{40}$$

Can be considered as a wave propagating along the interface (here in x_1 direction) with an amplitude depending on x_3:

$$\vec{a}_{\mathrm{eff}} = e^{\pm\omega s_1 S x_3}\,\vec{a} \tag{41}$$

Only the version where $\vec{a}_{\mathrm{eff}}$ decreases exponentially with distance from the interface ($|x_3|$) makes sense:

$$\vec{a}_{\mathrm{eff}} = e^{-\omega s_1 S |x_3|}\,\vec{a} \tag{42}$$

Respective waves are called interface waves.

The Depth of Penetration of Interface Waves

$$\vec{a}_{\text{eff}} \;=\; e^{-\omega s_1 S |x_3|}\,\vec{a} \;=\; e^{-\frac{|x_3|}{d}}\,\vec{a} \tag{43}$$

with the depth of penetration

$$d \;=\; \frac{1}{\omega s_1 S} \;=\; \frac{L}{2\pi S} \tag{44}$$

and the wavelength $L = \frac{2\pi}{\omega s_1}$.

- $d \to \infty$ (plane wave propagating along the surface) if the wave is only slightly too slow for the medium ($s_1 \to |\vec{s}|$).
- $d \to \frac{L}{2\pi}$ if the wave is much too slow for the medium ($s_1 \gg |\vec{s}|$).

Particle Orbits of Interface Waves

Examples of particle orbits for an incident SV wave at the crust-mantle boundary: $\alpha = 20^\circ$, $\alpha = 30^\circ$, $\alpha = 40^\circ$, $\alpha = 70^\circ$

- Particles move on elliptical orbits.
- Prograde rotation in the lower halfspace; retrograde rotation in the upper halfspace.

Particle Orbits of Interface Waves

For the P-wave, $\vec{a}$ must be parallel to $\vec{s}$:

$$\vec{a} \;\propto\; \vec{s} \;=\; \begin{pmatrix} s_1 \\ 0 \\ iSs_1 \end{pmatrix} \;\propto\; \begin{pmatrix} 1 \\ 0 \\ iS \end{pmatrix} \tag{45}$$

(only in the lower halfspace, $-i$ in the upper halfspace)

$$\Downarrow$$

$$\vec{u}(\vec{0}, t) \;=\; e^{i\omega t}\,\vec{a} \;\propto\; e^{i\omega t} \begin{pmatrix} 1 \\ 0 \\ iS \end{pmatrix} \tag{46}$$

$$\propto \; \begin{pmatrix} \cos(\omega t) \\ 0 \\ -S\sin(\omega t) \end{pmatrix} + i \begin{pmatrix} \sin(\omega t) \\ 0 \\ S\cos(\omega t) \end{pmatrix} \tag{47}$$

- Vertical ellipses with aspect ratio $\frac{1}{S}$; circles for $S \rightarrow 1$ ($s_1 \gg |\vec{s}|$).
- Prograde rotation (retrograde in the upper halfspace)

Particle Orbits of Interface Waves

For the SV-wave, $\vec{a}$ must be normal to $\vec{s}$:

$$\vec{a} \;\propto\; \begin{pmatrix} -iSs_1 \\ 0 \\ s_1 \end{pmatrix} \;\propto\; \begin{pmatrix} 1 \\ 0 \\ \frac{i}{S} \end{pmatrix} \tag{48}$$

$$\Downarrow$$

$$\vec{u}(\vec{0}, t) \;=\; e^{i\omega t}\,\vec{a} \;\propto\; e^{i\omega t} \begin{pmatrix} 1 \\ 0 \\ \frac{i}{S} \end{pmatrix} \tag{49}$$

$$\propto\; \begin{pmatrix} \cos(\omega t) \\ 0 \\ -\frac{1}{S}\sin(\omega t) \end{pmatrix} + i \begin{pmatrix} \sin(\omega t) \\ 0 \\ \frac{1}{S}\cos(\omega t) \end{pmatrix} \tag{50}$$

- Vertical ellipses with aspect ratio $\frac{1}{S}$; circles for $S \to 1$ ($s_1 \gg |\vec{s}|$).
- Prograde rotation (retrograde in the upper halfspace)

Surface Waves at a Free Surface

Interface waves: driven by plane waves (incident, reflected, refracted)

Surface waves: living on their own at a free surface (interface to air)

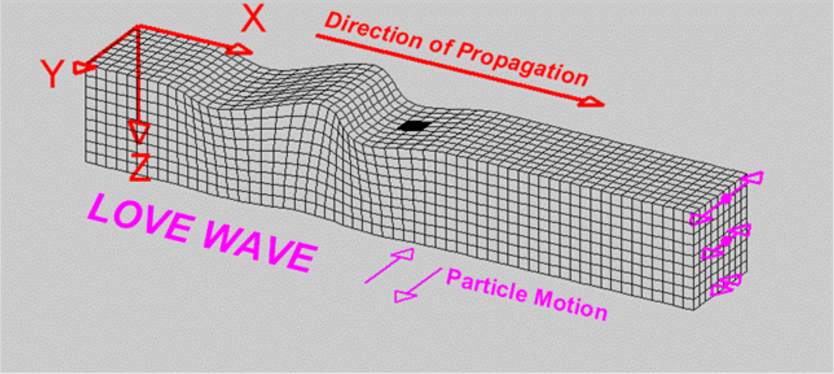
$$\Downarrow$$

Two fundamental types of surface waves in a semi-infinite halfspace:

Rayleigh wave, named after J. W. Strutt (later 3. Lord Rayleigh)

Love wave, named after A. E. H. Love

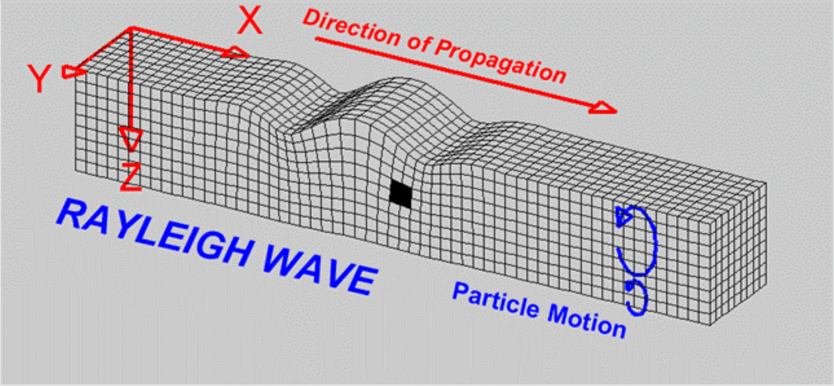
The Love Wave



Source: L. Braile, Purdue University

SH interface wave; not possible in a homogeneous halfspace because $\vec{\sigma}_{\text{int}} \neq \vec{0}$ for $a_2 \neq 0$ (Eq. 34)

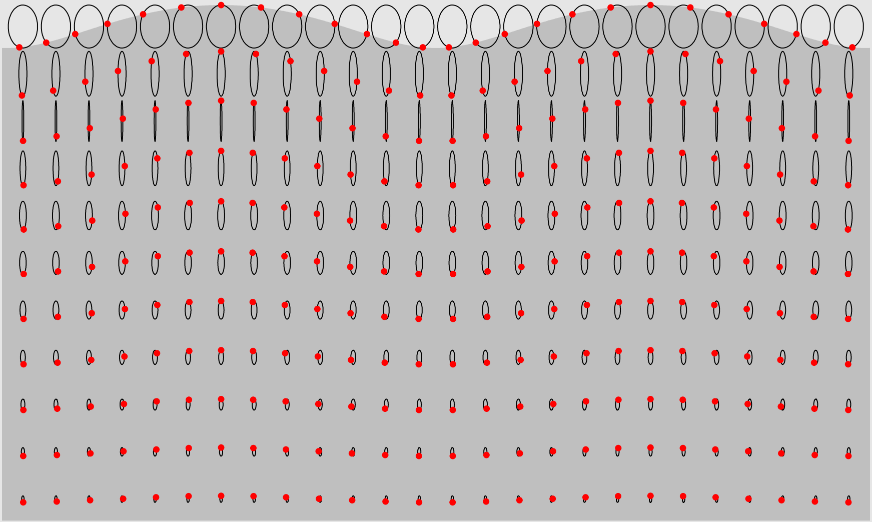
The Rayleigh Wave



Source: L. Braile, Purdue University

Specific superposition of P and SH interface wave

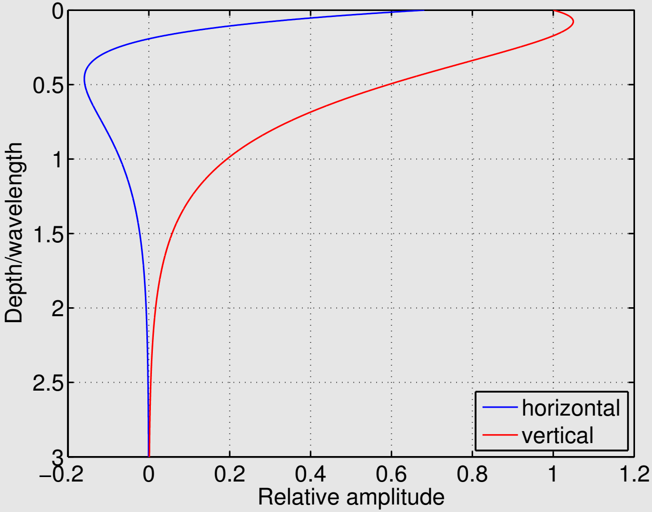
The Rayleigh Wave in a Homogeneous Poisson Solid ($\lambda = \mu$)



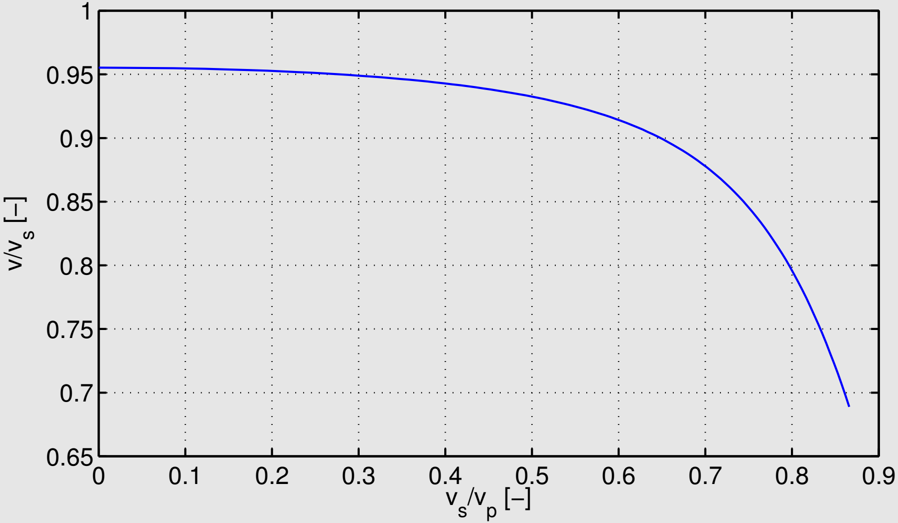
The Rayleigh Wave in a Homogeneous Poisson Solid ($\lambda = \mu$)

- Retrograde particle motion on elliptical orbits at the surface.
- Prograde particle motion on elliptical orbits at greater depth.
- Velocity $v \approx 0.92\, v_s$.

The Rayleigh Wave in a Homogeneous Poisson Solid ($\lambda = \mu$)



The Rayleigh Wave in a Homogeneous Halfspace



Surface Waves in Inhomogeneous Media

Assume that ρ, λ, and μ depend on the vertical coordinate x_3, and generalize

$$\vec{u}(\vec{x},t) \;=\; e^{i\omega(t-s_1x_1)}\left(e^{k_1S_px_3}a_p\begin{pmatrix}1\\0\\iS_p\end{pmatrix}+e^{k_1S_sx_3}a_s\begin{pmatrix}1\\0\\\frac{i}{S_s}\end{pmatrix}\right) \tag{51}$$

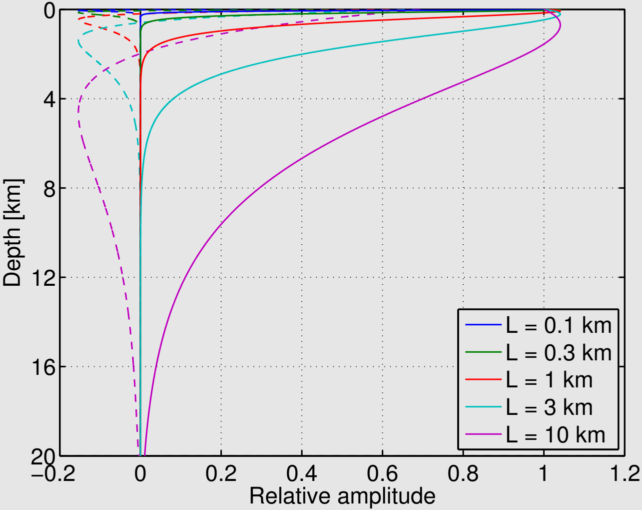
to

$$\vec{u}(\vec{x},t) \;=\; e^{i\omega(t-s_1x_1)}\,\vec{a}(x_3) \tag{52}$$

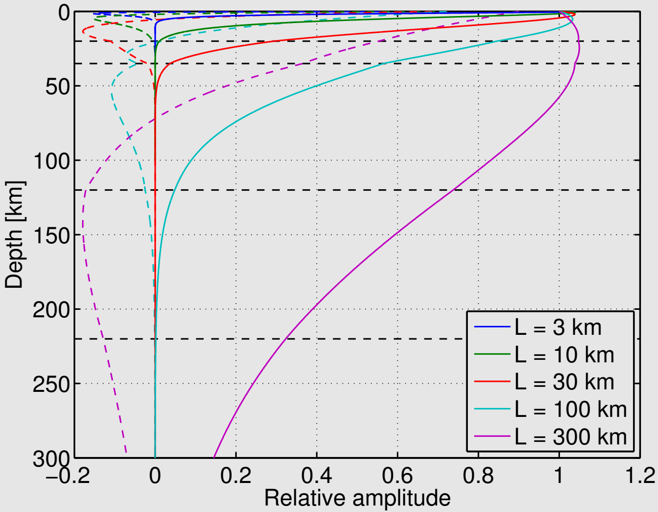
$$\Downarrow$$

- Differential equations (eigenvalue problem) for $a_1(x_3)$, $a_2(x_3)$, $a_3(x_3)$.
- $a_1(x_3)$ and $a_3(x_3)$ are coupled (➔ Rayleigh wave), $a_2(x_3)$ is independent of $a_1(x_3)$ and $a_3(x_3)$ (➔ Love wave).
- Must be solved numerically.

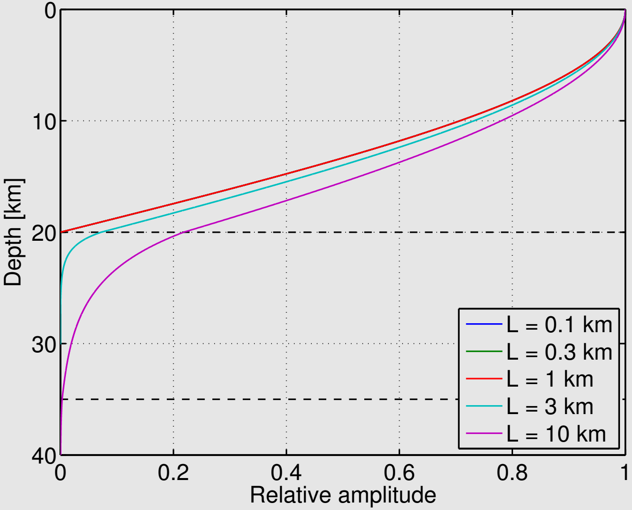
Rayleigh Waves in Typical Continental Subsurface (PEM)



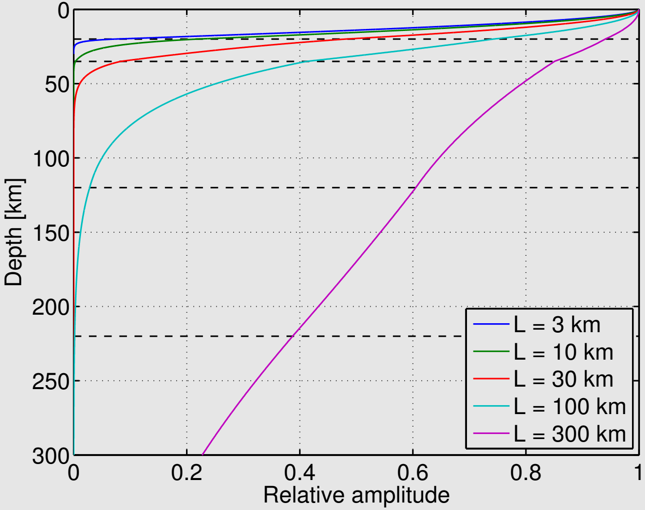
Rayleigh Waves in Typical Continental Subsurface (PEM)



Love Waves in Typical Continental Subsurface (PEM)



Love Waves in Typical Continental Subsurface (PEM)



Main Differences Between Body Waves and Surface Waves

- Decrease with the distance from the hypocenter/epicenter r:

	Energy flux density	Amplitude
body waves	$\propto \frac{1}{r^2}$	$\propto \frac{1}{r}$
surface waves	$\propto \frac{1}{r}$	$\propto \frac{1}{\sqrt{r}}$

$\Downarrow$

Surface waves have a longer range than body waves.

- Velocity of harmonic surface waves depends on the wavelength (dispersion).

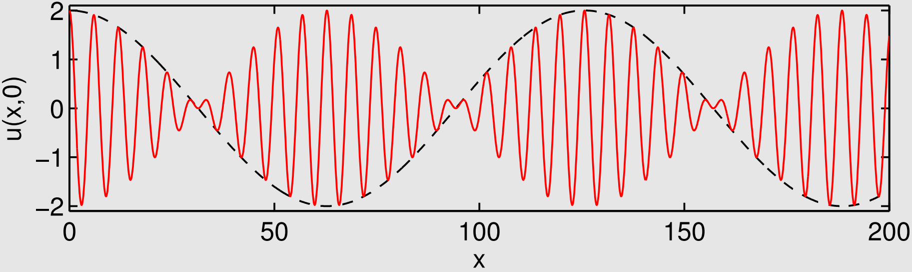
Dispersion

Simplest situation: superposition of two harmonic waves with the same amplitude ($=1$), but different frequencies in 1D:

$$u(x,t) \;=\; e^{i\omega_1(t-s_1x)} + e^{i\omega_2(t-s_2x)} \;=\; e^{i(\omega_1 t-k_1x)} + e^{i(\omega_2 t-k_2x)} \tag{53}$$

with $k_1 = \omega_1 s_1$, $k_2 = \omega_2 s_2$.

Example: $k_1 = 10$, $k_2 = 11$



Dispersion

$$
\begin{aligned}
u(x,t) &= e^{i\omega_1(t-s_1x)} + e^{i\omega_2(t-s_2x)} = e^{i(\omega_1 t-k_1x)} + e^{i(\omega_2 t-k_2x)} && (54)\\
&= e^{i\left(\overline{\omega}t+\frac{\omega_1-\omega_2}{2}t-\overline{k}x-\frac{k_1-k_2}{2}x\right)} + e^{i\left(\overline{\omega}t-\frac{\omega_1-\omega_2}{2}t-\overline{k}x+\frac{k_1-k_2}{2}x\right)} && (55)\\
&= e^{i\left(\overline{\omega}t-\overline{k}x\right)}\left(e^{i\left(\frac{\omega_1-\omega_2}{2}t-\frac{k_1-k_2}{2}x\right)} + e^{i\left(-\frac{\omega_1-\omega_2}{2}t+\frac{k_1-k_2}{2}x\right)}\right) && (56)\\
&= e^{i\left(\overline{\omega}t-\overline{k}x\right)} 2\cos\left(\frac{\omega_1-\omega_2}{2}t-\frac{k_1-k_2}{2}x\right) && (57)\\
&= e^{i\overline{\omega}\left(t-\frac{\overline{k}}{\overline{\omega}}x\right)} 2\cos\left(\frac{\omega_1-\omega_2}{2}\left(t-\frac{k_1-k_2}{\omega_1-\omega_2}x\right)\right) && (58)
\end{aligned}
$$

with $\overline{\omega}=\frac{\omega_1+\omega_2}{2}$, $\overline{k}=\frac{k_1+k_2}{2}$.

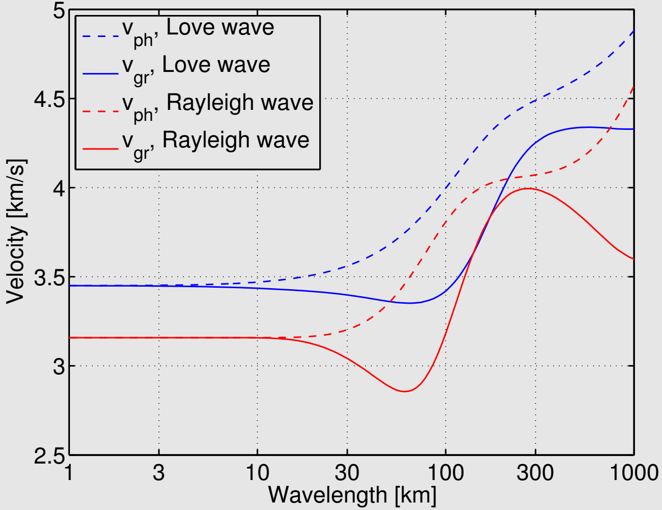
Dispersion

For $\omega_1 \approx \omega_2$, $k_1 \approx k_2$:

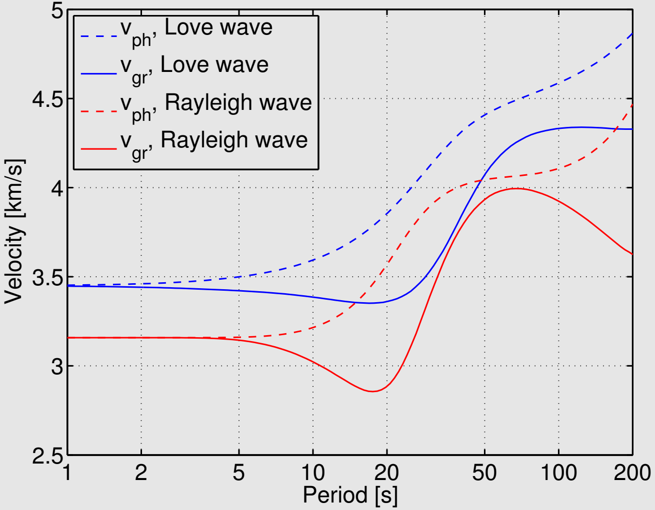
- High-frequency oscillation with an angular frequency $\overline{\omega}$ propagating with the phase slowness $s_{\mathrm{ph}} = \frac{\overline{k}}{\overline{\omega}}$
- Low-frequency oscillation of the amplitude with an angular frequency $\frac{\omega_1 - \omega_2}{2}$ propagating with the group slowness

$$s_{\mathrm{gr}} = \frac{k_1 - k_2}{\omega_1 - \omega_2} \rightarrow \frac{dk}{d\omega} \quad \text{for} \quad \omega_1 - \omega_2 \rightarrow 0 \tag{59}$$

Velocities of Surface Waves in Typical Continental Subsurface (PEM)



Velocities of Surface Waves in Typical Continental Subsurface (PEM)



Dispersion of Rayleigh Waves in Typical Continental Subsurface

Consider wave propagation in 1D with a Gaussian peak

$$u(0,t) \;=\; e^{-\frac{t^2}{2\sigma^2}} \tag{60}$$

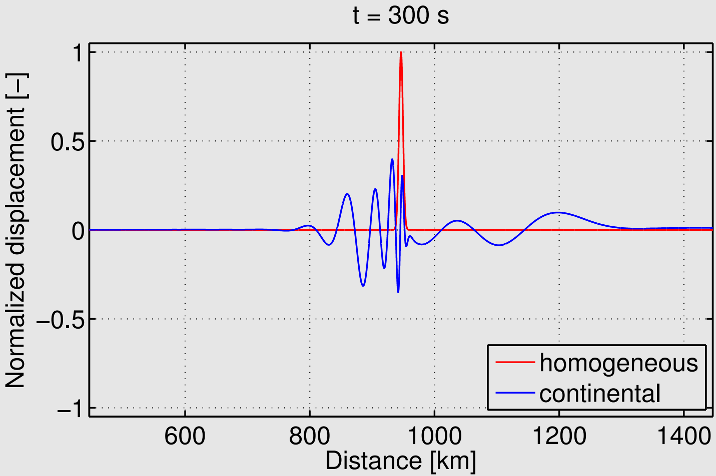
as a source ($x = 0$); alternatively

$$u(x,0) \;=\; e^{-\frac{(sx)^2}{2\sigma^2}} \tag{61}$$

as an initial condition.

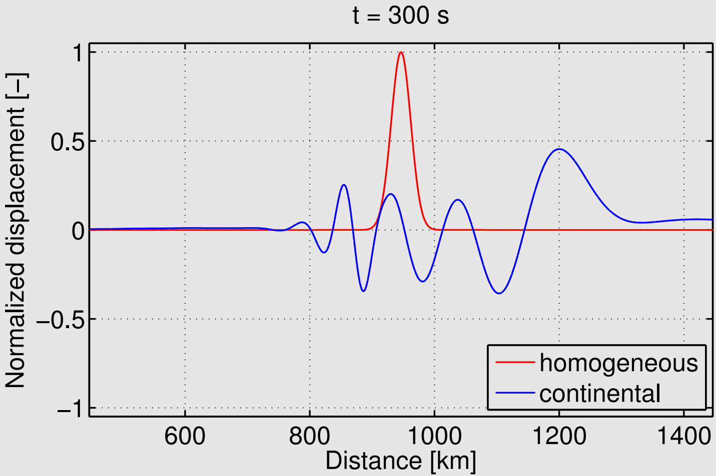
Dispersion of Rayleigh Waves in Typical Continental Subsurface

Example 1: short pulse; $\sigma = 1\,\mathrm{s}$

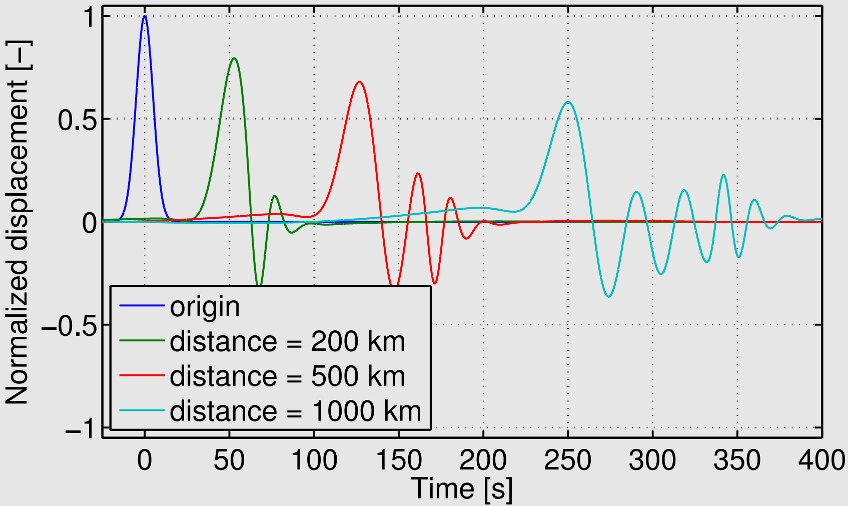


Dispersion of Rayleigh Waves in Typical Continental Subsurface

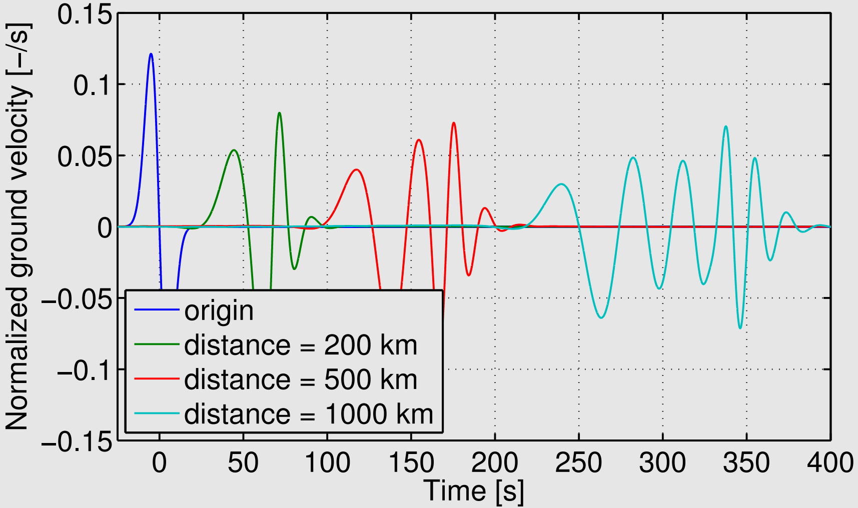
Example 2: longer pulse; $\sigma = 5\,\mathrm{s}$



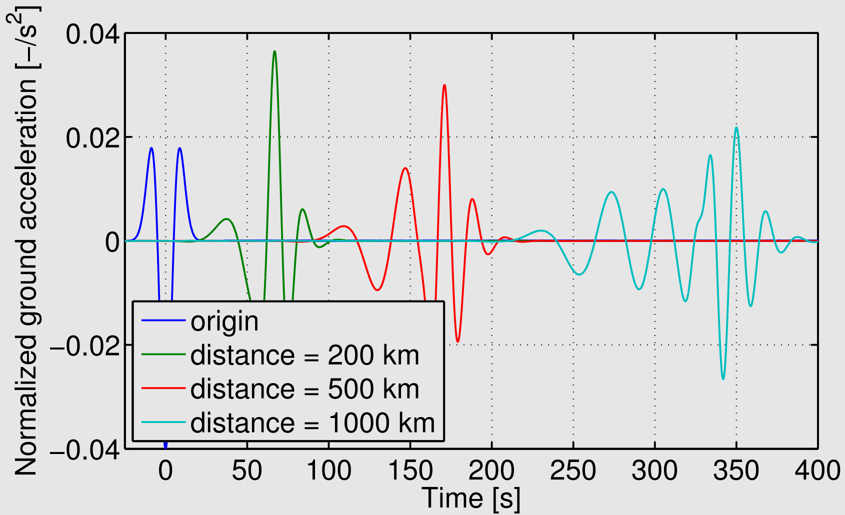
Dispersion of Rayleigh Waves in Typical Continental Subsurface



Dispersion of Rayleigh Waves in Typical Continental Subsurface



Dispersion of Rayleigh Waves in Typical Continental Subsurface



The Point-Force Solution

- Infinite, homogeneous medium with parameters ρ, λ, and μ (like plane wave consideration).
- Assume that a given force $\vec{F}(t)$ acts at the origin ($\vec{x} = \vec{0}$).

Respective solution of the Navier-Cauchy equations:

$$\vec{u}_\mathrm{f}(\vec{x}, t) = \frac{s_p^2}{4\pi\rho r}\mathbf{P}\vec{F}(t - s_p r) + \frac{s_s^2}{4\pi\rho r}(\mathbf{1} - \mathbf{P})\vec{F}(t - s_s r) \tag{62}$$

$$+\frac{1}{4\pi\rho r^3}(3\mathbf{P} - \mathbf{1})\int_{s_p r}^{s_s r} \tau\vec{F}(t - \tau)d\tau \tag{63}$$

where

$$r = |\vec{x}|$$

$$\mathbf{P} = \vec{e}\,\vec{e}^{\,T} = \text{projection on radial direction}, \quad \vec{e} = \frac{\vec{x}}{r}$$

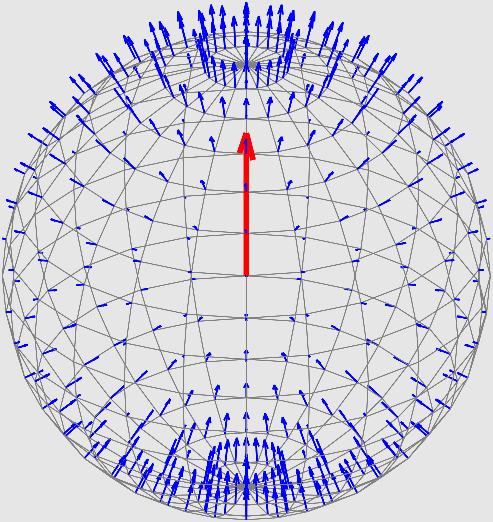
The Point-Force Solution

Spatial pattern of the first term,

$$\mathbf{P}\vec{F},$$

for

$$\vec{F} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$



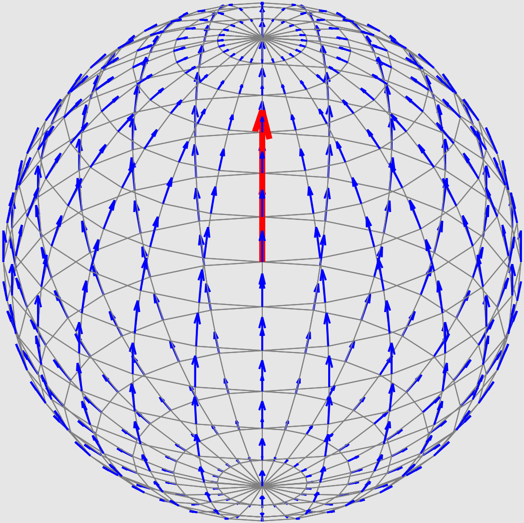
The Point-Force Solution

Spatial pattern of the second term,

$$(\mathbf{1} - \mathbf{P})\,\vec{F},$$

for

$$\vec{F} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$



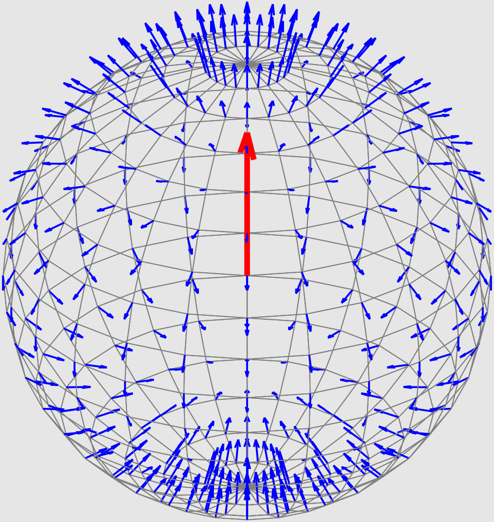
The Point-Force Solution

Spatial pattern of the third term,

$$(3\mathbf{P} - \mathbf{1})\,\vec{F}, \tag{64}$$

for

$$\vec{F} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$



Force Couples

Solution for a single point force causes an overall displacement in direction of the force.

$\Downarrow$

not possible

$\Downarrow$

Consider a couple of opposite forces $\vec{F}$ and $-\vec{F}$ displaced by a small vector $\vec{a}$ (at $\frac{\vec{a}}{2}$ and $-\frac{\vec{a}}{2}$).

$\Downarrow$

$$\vec{u}(\vec{x}, t) \;=\; \vec{u}_{\mathrm{f}}(\vec{x} - \tfrac{\vec{a}}{2}, t) - \vec{u}_{\mathrm{f}}(\vec{x} + \tfrac{\vec{a}}{2}, t) \tag{65}$$

The Seismic Moment Tensor

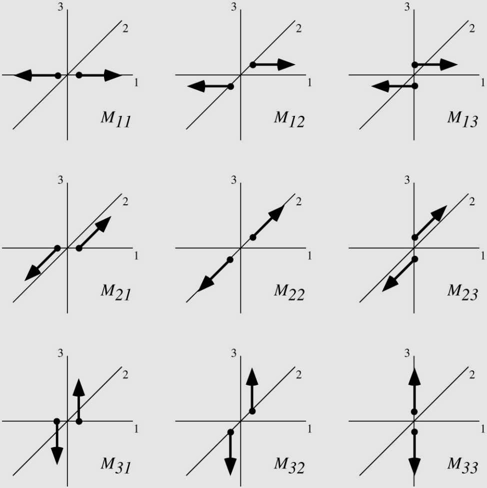
Approximation in the limit $\vec{a} \rightarrow \vec{0}$:

$$
\begin{aligned}
\vec{u}(\vec{x},t) &\approx -\nabla \vec{u}_{\mathrm{f}}(\vec{x},t)\vec{a} && (66)\\
&= -\mathrm{div}\left(\vec{u}_{\mathrm{f}}(\vec{x},t)\,\vec{a}^{T}\right) && (67)\\
&= -\mathrm{div}\left(\frac{s_p^2}{4\pi\rho r}\mathbf{P}\mathbf{M}(t-s_p r) + \frac{s_s^2}{4\pi\rho r}\left(\mathbf{1}-\mathbf{P}\right)\mathbf{M}(t-s_s r)\right.\\
&\qquad \left. + \frac{1}{4\pi\rho r^3}\left(3\mathbf{P}-\mathbf{1}\right)\int_{s_p r}^{s_s r} \tau \mathbf{M}(t-\tau)d\tau\right) && (68)
\end{aligned}
$$

with the seismic moment tensor (centroid moment tensor, CMT)

$$
\mathbf{M}(t) = \vec{F}(t)\,\vec{a}^{T} \quad [Nm] \qquad (69)
$$

Components of the Seismic Moment Tensor



Source: Shearer, Introduction to Seismology

Symmetry of the Seismic Moment Tensor

Nondiagonal components of **M** cause an overall rotation.

$\Downarrow$

not possible

$\Downarrow$

M must be symmetric: $M_{ji} = M_{ij}$.

Far-Field Waves

$$\vec{u}(\vec{x},t) = -\mathrm{div}\left(\frac{s_p^2}{4\pi\rho r}\mathbf{P}\mathbf{M}(t-s_p r)+\frac{s_s^2}{4\pi\rho r}\left(\mathbf{1}-\mathbf{P}\right)\mathbf{M}(t-s_s r)\right.$$
$$\left.+\frac{1}{4\pi\rho r^3}\left(3\mathbf{P}-\mathbf{1}\right)\int_{s_p r}^{s_s r}\tau\mathbf{M}(t-\tau)d\tau\right) \qquad (70)$$

contains terms proportional to $\frac{1}{r}$, $\frac{1}{r^2}$, $\frac{1}{r^3}$, ...

$\Downarrow$

Terms proportional to $\frac{1}{r}$ dominate at great distances.

Far-Field Waves

Consider only terms proportional to $\frac{1}{r}$:

$$\begin{aligned}
\vec{u}(\vec{x}, t) &\approx -\frac{s_p^2}{4\pi\rho r}\mathbf{P}\,\mathrm{div}\,(\mathbf{M}(t - s_p r)) - \frac{s_s^2}{4\pi\rho r}(\mathbf{1}-\mathbf{P})\,\mathrm{div}\,(\mathbf{M}(t - s_s r)) \qquad (71) \\
&= \underbrace{\frac{s_p^3}{4\pi\rho r}\mathbf{P}\,\dot{\mathbf{M}}(t - s_p r)\vec{e}}_{\uparrow\ \text{P-wave}} + \underbrace{\frac{s_s^3}{4\pi\rho r}(\mathbf{1}-\mathbf{P})\,\dot{\mathbf{M}}(t - s_s r)\vec{e}}_{\uparrow\ \text{S-wave}} \qquad (72)
\end{aligned}$$

with

$$\dot{\mathbf{M}}(t) = \frac{d}{dt}\mathbf{M}(t) \qquad (73)$$

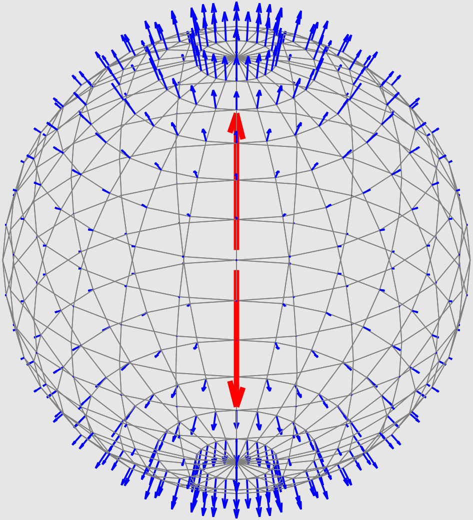
Far-Field Waves

P-wave radiation pattern

$$\mathbf{P}\dot{\mathbf{M}}\vec{e}$$

for

$$\dot{\mathbf{M}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



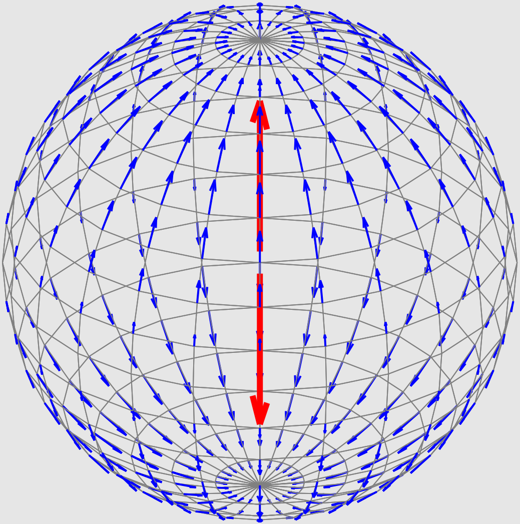
Far-Field Waves

S-wave radiation pattern

$$(\mathbf{1}-\mathbf{P})\,\dot{\mathbf{M}}\vec{e}$$

for

$$\dot{\mathbf{M}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



Far-Field Waves

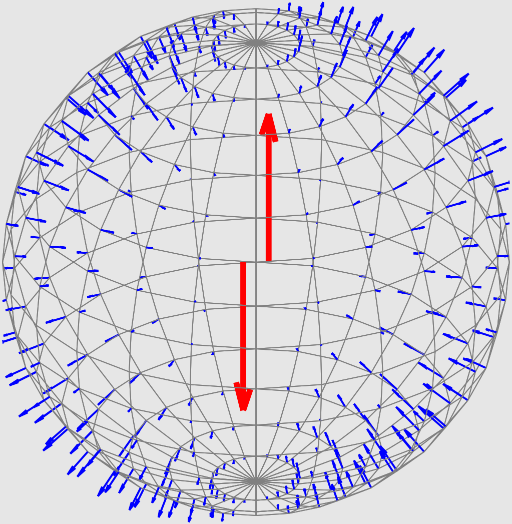
P-wave radiation pattern

$$\mathbf{P}\dot{\mathbf{M}}\vec{e}$$

for

$$\dot{\mathbf{M}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

(not allowed!)



Far-Field Waves

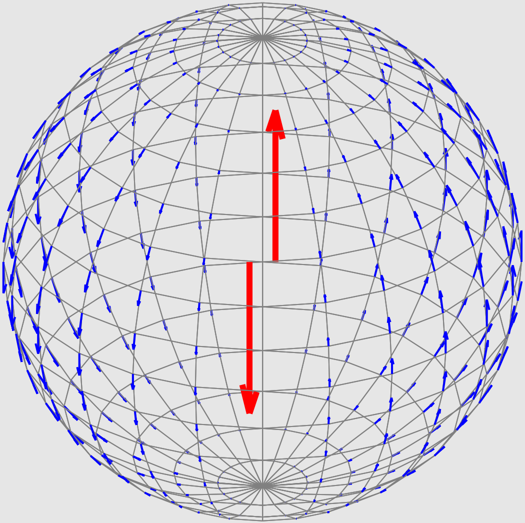
S-wave radiation pattern

$$(\mathbf{1} - \mathbf{P})\,\dot{\mathbf{M}}\vec{e}$$

for

$$\dot{\mathbf{M}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

(not allowed!)



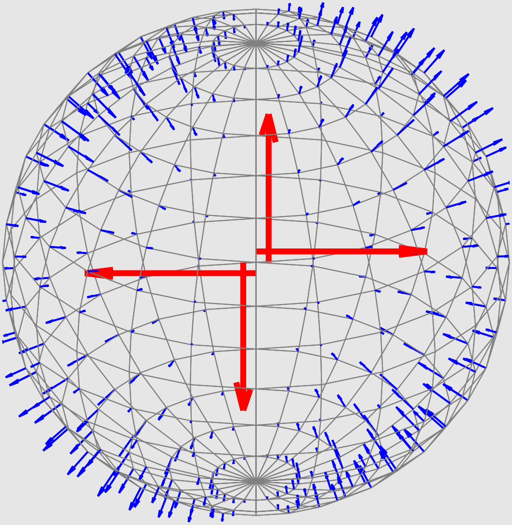
Far-Field Waves

P-wave radiation pattern

$$\mathbf{P}\dot{\mathbf{M}}\vec{e}$$

for

$$\dot{\mathbf{M}} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$



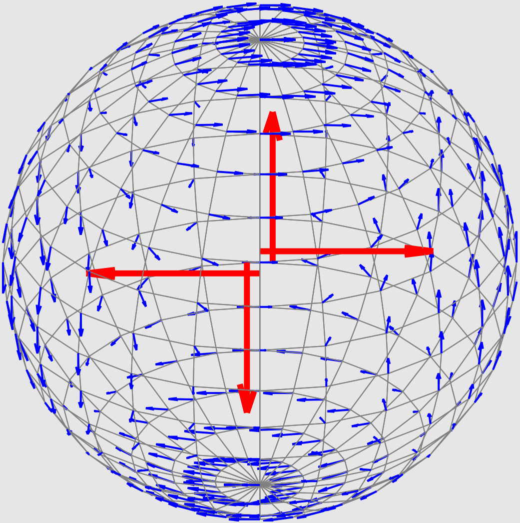
Far-Field Waves

S-wave radiation pattern

$$(\mathbf{1} - \mathbf{P})\,\dot{\mathbf{M}}\vec{e}$$

for

$$\dot{\mathbf{M}} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$



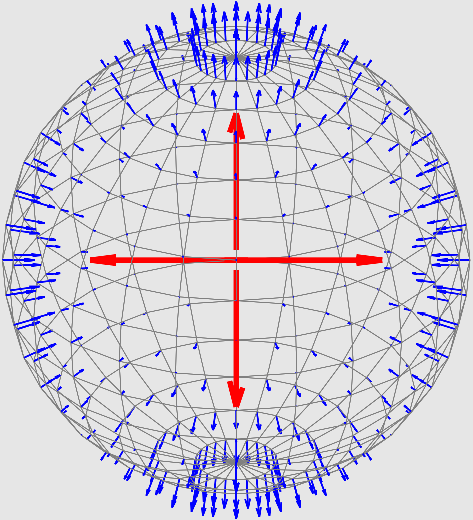
Far-Field Waves

P-wave radiation pattern

$$\mathbf{P}\dot{\mathbf{M}}\vec{e}$$

for

$$\dot{\mathbf{M}} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



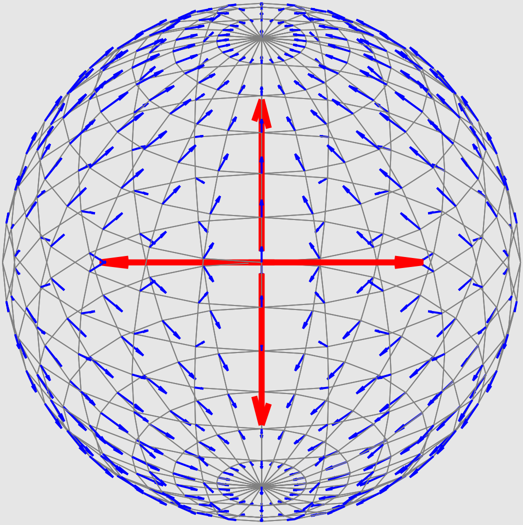
Far-Field Waves

S-wave radiation pattern

$$(\mathbf{1} - \mathbf{P})\,\dot{\mathbf{M}}\vec{e}$$

for

$$\dot{\mathbf{M}} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



The Scalar Seismic Moment

If

$$\mathbf{M} = \begin{pmatrix} 0 & 0 & M_0 \\ 0 & 0 & 0 \\ M_0 & 0 & 0 \end{pmatrix} \quad \text{or} \quad \mathbf{M} = \begin{pmatrix} -M_0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & M_0 \end{pmatrix} \tag{74}$$

(or similar), M_0 is called (scalar) seismic moment.

In general:

$$M_0 = \frac{M_1 - M_3}{2} \tag{75}$$

where M_1, M_2, and M_3 are the eigenvalues of $\mathbf{M}$ in descending order.

The Scalar Seismic Moment

Alternative interpretation of the seismic moment:

$$M_0 = \mu A \overline{u} \tag{76}$$

where

A = size of the rupture area [m^2]

$\overline{u}$ = mean displacement along the rupture area [m]

Amplitudes of Body Waves

P-wave displacement at large distances:

$$\vec{u}_p(\vec{x}, t) = \frac{s_p^3}{4\pi\rho r} \mathbf{P}\,\dot{\mathbf{M}}(t - s_p r)\vec{e} \tag{77}$$

Maximum displacement occurs in the directions of the first and third principal axes of $\dot{\mathbf{M}}$:

$$|\vec{u}_p|_{\max} = \frac{s_p^3}{4\pi\rho r} |\dot{M}_0|_{\max} \tag{78}$$

Amplitudes of Body Waves

S-wave displacement at large distances:

$$\vec{u}_s(\vec{x}, t) \;=\; \frac{s_s^3}{4\pi\rho r}\,(\mathbf{1} - \mathbf{P})\,\dot{\mathbf{M}}(t - s_s r)\vec{e} \tag{79}$$

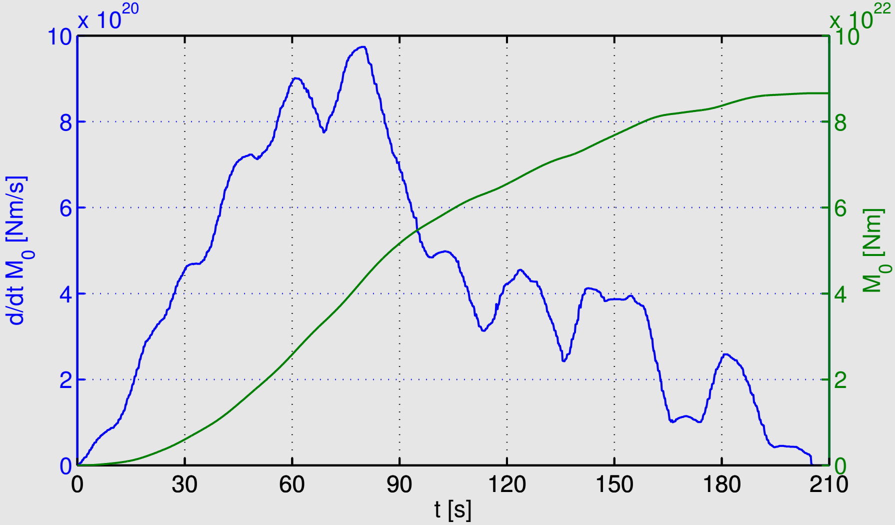
Maximum displacement occurs in the directions 45° between the first and third principal axis of $\dot{\mathbf{M}}$:

$$|\vec{u}_s|_{\max} \;=\; \frac{s_s^3}{4\pi\rho r}|\dot{M}_0|_{\max} \tag{80}$$

$$\Downarrow$$

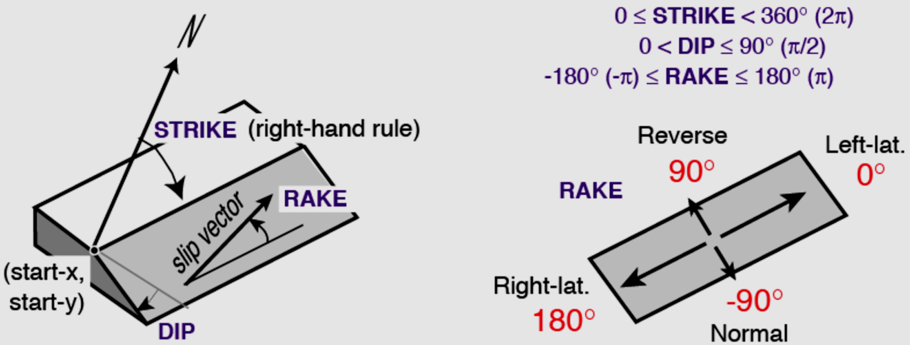
$$\frac{|\vec{u}_s|_{\max}}{|\vec{u}_p|_{\max}} \;=\; \frac{s_s^3}{s_p^3} \;=\; \left(\frac{v_p}{v_s}\right)^3 \;\approx\; 5 \tag{81}$$

Seismic Moment vs. Moment Rate for the Alaska 1964 Earthquake



The Seismic Moment Tensor for a Double Force Couple

Definition of strike ϕ, dip δ, and rake λ according to Aki and Richards (1980)



Source: Toda et al., Coulomb 3.3 User Guide

The Seismic Moment Tensor for a Double Force Couple

Step 1: Start with a force couple in x_2 direction displaced in x_3 direction:

$$\vec{F} = \begin{pmatrix} 0 \\ F \\ 0 \end{pmatrix}, \quad \vec{a} = \begin{pmatrix} 0 \\ 0 \\ a \end{pmatrix} \tag{82}$$

Step 2: Rotate $\vec{F}$ and $\vec{a}$ counterclockwise by the rake angle λ in the x_1-x_2 plane:

$$\vec{F}_\lambda = \mathbf{R}_\lambda \vec{F}, \quad \vec{a}_\lambda = \mathbf{R}_\lambda \vec{a} = \vec{a} \tag{83}$$

with

$$\mathbf{R}_\lambda = \begin{pmatrix} \cos\lambda & -\sin\lambda & 0 \\ \sin\lambda & cos\lambda & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{84}$$

The Seismic Moment Tensor for a Double Force Couple

Step 3: Rotate $\vec{F}_\lambda$ and $\vec{a}_\lambda$ clockwise by the dip angle δ in the x_1-x_3 plane:

$$\vec{F}_{\lambda\delta} = \mathbf{R}_\delta \vec{F}_\lambda = \mathbf{R}_\delta \mathbf{R}_\lambda \vec{F}, \quad \vec{a}_{\lambda\delta} = \mathbf{R}_\delta \vec{a}_\lambda = \mathbf{R}_\delta \mathbf{R}_\lambda \vec{a} \tag{85}$$

with

$$\mathbf{R}_\delta = \begin{pmatrix} \cos\delta & 0 & \sin\delta \\ 0 & 1 & 0 \\ -\sin\delta & 0 & cos\delta \end{pmatrix} \tag{86}$$

The Seismic Moment Tensor for a Double Force Couple

Step 4: Rotate $\vec{F}_{\lambda\delta}$ and $\vec{a}_{\lambda\delta}$ clockwise by the strike angle ϕ in the x_1-x_2 plane:

$$\vec{F}_{\lambda\delta\phi} = \mathbf{R}_\phi \vec{F}_{\lambda\delta} = \mathbf{R}_\phi \mathbf{R}_\delta \mathbf{R}_\lambda \vec{F}, \quad \vec{a}_{\lambda\delta\phi} = \mathbf{R}_\phi \vec{a}_{\lambda\delta} = \mathbf{R}_\phi \mathbf{R}_\delta \mathbf{R}_\lambda \vec{a} \quad (87)$$

with

$$\mathbf{R}_\phi = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (88)$$

$$\Downarrow$$

$$\vec{F}_{\lambda\delta\phi} = \mathbf{R}\vec{F}, \quad \vec{a}_{\lambda\delta\phi} = \mathbf{R}\vec{a} \quad \text{with} \quad \mathbf{R} = \mathbf{R}_\phi \mathbf{R}_\delta \mathbf{R}_\lambda \quad (89)$$

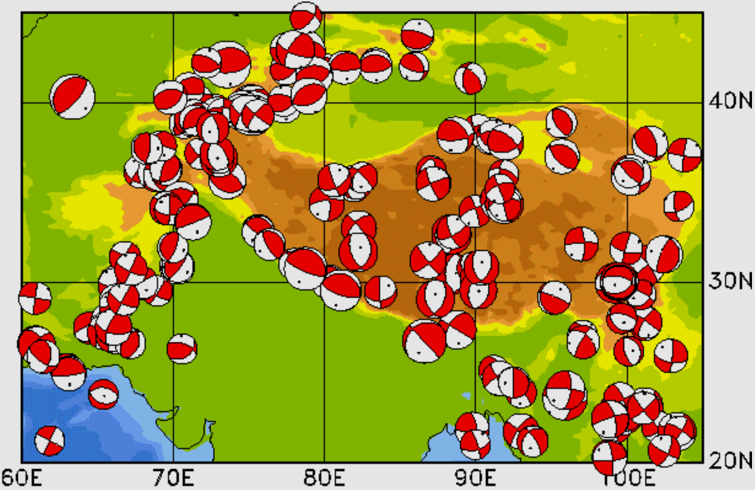
The Seismic Moment Tensor for a Double Force Couple

$$\Downarrow$$

$$\begin{aligned}
\mathbf{M} &= \vec{F}_{\lambda\delta\phi}\,\vec{a}^{T}_{\lambda\delta\phi} + \vec{a}_{\lambda\delta\phi}\,\vec{F}^{T}_{\lambda\delta\phi} = \left(\mathbf{R}\vec{F}\right)(\mathbf{R}\vec{a})^{T} + (\mathbf{R}\vec{a})\left(\mathbf{R}\vec{F}\right)^{T} && (90)\\
&= \mathbf{R}\left(\vec{F}\vec{a}^{T} + \vec{a}\vec{F}^{T}\right)\mathbf{R}^{T} && (91)\\
&= \mathbf{R}\left(\begin{pmatrix}0\\F\\0\end{pmatrix}\begin{pmatrix}0\\0\\a\end{pmatrix}^{T} + \begin{pmatrix}0\\0\\a\end{pmatrix}\begin{pmatrix}0\\F\\0\end{pmatrix}^{T}\right)\mathbf{R}^{T} && (92)\\
&= M_0\,\mathbf{R}\begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix}\mathbf{R}^{T} && (93)
\end{aligned}$$

with $M_0 = Fa$

Beachball Plots



Source: Earthquake-Report.com

Beachball Plots

Basic assumption: $\dot{\mathbf{M}}(t)$ has the same shape as $\mathbf{M}(t)$,

$$\mathbf{M}(t) \;=\; f(t)\,\mathbf{M}, \quad \dot{\mathbf{M}}(t) \;=\; \dot{f}(t)\,\mathbf{M} \tag{94}$$

where $\mathbf{M}$ is the total seismic moment, and $f(t)$ increases from 0 to 1.

P-wave radiation pattern:

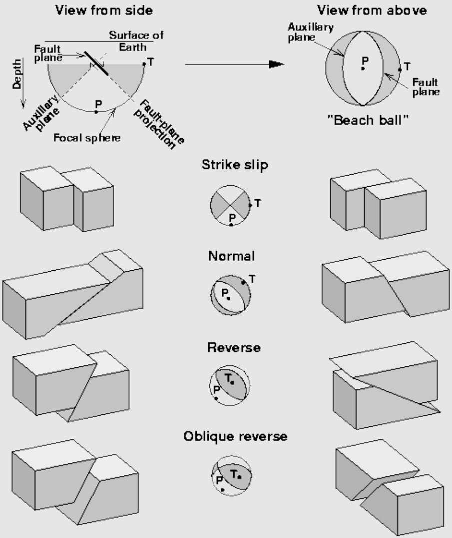
$$\vec{u} \;\propto\; \mathbf{PM}\vec{e} \;=\; ((\mathbf{M}\vec{e})\cdot\vec{e})\,\vec{e} \tag{95}$$

$$\Downarrow$$

P-wave arrives with compression first if $(\mathbf{M}\vec{e})\cdot\vec{e} > 0$ and with dilatation if $(\mathbf{M}\vec{e})\cdot\vec{e} < 0$.

Beachball Plots

- Directions where the P-wave arrives first with compression $((\mathbf{M}\vec{e}) \cdot \vec{e} > 0)$ are colored.
- Directions where the P-wave arrives first with dilatation $(\mathbf{M}\vec{e}) \cdot \vec{e} < 0$ are left white.
- Projection of the lower half of the sphere is plotted (sometimes stereographic projection, but mostly equal-area projection).



Source: US Geological Survey

Beachball Plots

If the eigenvalues of $\mathbf{M}$ are M_0, 0, and $-M_0$, the sphere consists of 4 equal quadrants.

Examples of beachball plots:

- normal fault ($\lambda = -90^\circ$) for different dip angles δ
- reverse fault ($\lambda = 90^\circ$) for different dip angles δ
- transform fault ($\lambda = 0^\circ$) for different dip angles δ
- fault dipping at $\delta = 45^\circ$ for different rake angles λ
- fault dipping at $\delta = 45^\circ$ for different rake angles λ with additional isotropic expansion

Intensity and Magnitude

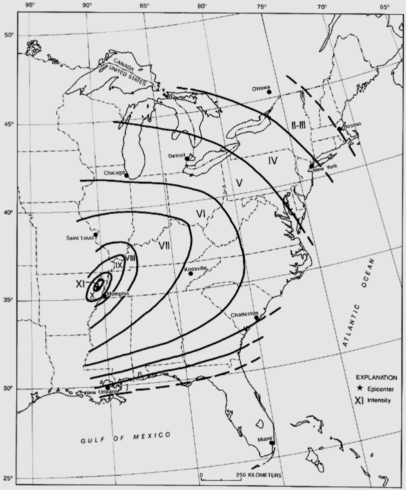
Intensity describes the severity of an earthquake in terms of its effects on the Earth's surface and on humans and their structures.

- Usually written as a Roman numeral.
- Goes back to a 12 level scale (originally 10) from I (not felt) to XII (total destruction) named after G. Mercalli (1850–1914).
- Several extensions / refinements: MCS (Mercalli-Cancani-Sieberg) scale, MWN (Mercalli-Wood-Neumann) scale, MSK scale (Medvedev, Sponheuer & Karnik, 1964), EMS-98 scale (European Macroseismic Scale, 2000).

Magnitude characterizes the size of an earthquake using measured values.

- Usually written as an Arabic numeral with one decimal digit.
- Several different magnitude definitions.
- Logarithmic scale.

Example of an Isoseismal Map of Earthquake Intensity



Source: USGS

General Definition of Earthquake Magnitude

If X is any physically measured property of an earthquake, e. g.

- total seismic moment M_0 or
- maximum ground displacement $|\vec{u}|_{\text{max}}$,

the corresponding earthquake magnitude is defined by

$$M_X \;=\; e \log_{10} \left(\frac{X}{X_0} \right) \tag{96}$$

where

$$\begin{array}{rcl} X_0 & = & \text{measured value for an earthquake of } M_X = 0 \text{ under the same conditions} \\ e & = & \text{factor used for making different magnitude definitions consistent (mostly } e = 1\text{)} \end{array}$$

General Definition of Earthquake Magnitude

If X is a property related to any point different from the earthquake focus, X_0 is a function of distance Δ and depth h (and other properties).

$$\Downarrow$$

$$M_X = e \log_{10}\left(\frac{X}{X_0(\Delta, h)}\right) = e \log_{10} X + \sigma(\Delta, h) \tag{97}$$

with the distance-depth correction function

$$\sigma(\Delta, h) = -e \log_{10} X_0(\Delta, h) \tag{98}$$

This only makes sense if the distance-depth dependence of X is independent of X itself.

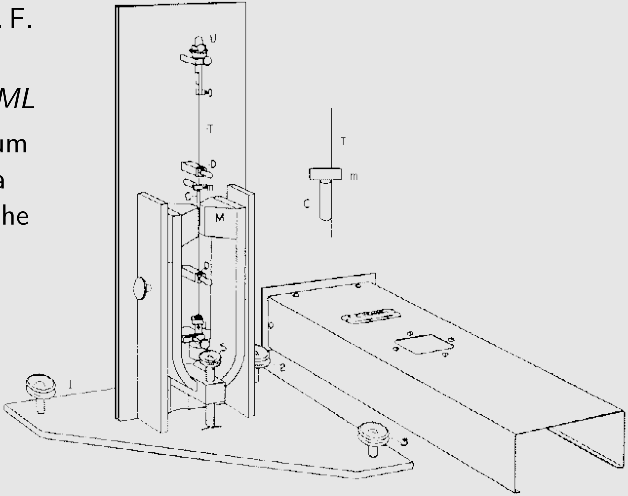
Upper und Lower Limits of Magnitude Scales

- All magnitude scales are from their definition open and both ends.
- Upper limits on Earth are introduced by geological constraints and by the process of wave propagation.
- Negative magnitudes are possible. The definition of zero magnitude is arbitrary and corresponds to what was detectable when the first magnitude definition (C. F. Richter, 1935) was introduced.

Earthquake Magnitude

The Local Magnitude (Richter Scale)

- Introduced by C. F. Richter in 1935.
- Symbol: M_L or ML
- X is the maximum amplitude A of a specific device, the Wood-Anderson seismometer.



Source: Southern California Earthquake Data Center

The Wood-Anderson Seismometer

- Oscillation by torsion of a wire
- Electromagnetic damping
- Natural period of $\approx 0.8\,\text{s}$ (frequency $f_0 = 1.25\,\text{Hz}$); close to the natural period of many building structures.

$$\Downarrow$$

Relevant for earthquake hazard.

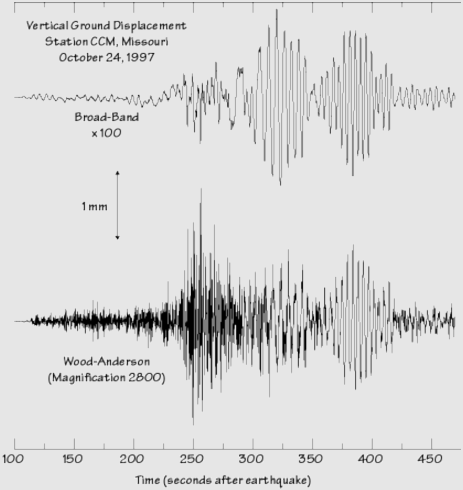
- Maximum magnification (record vs. ground displacelemt) of ≈ 2080 at f_0; sometimes a wrong value of 2800 was assumed.

$$\Downarrow$$

Local magnitudes derived from synthesized seismograms were too high for some time.

Earthquake Magnitude

The Wood-Anderson Seismometer



Source: C. J. Ammon, Pennsylvania State University

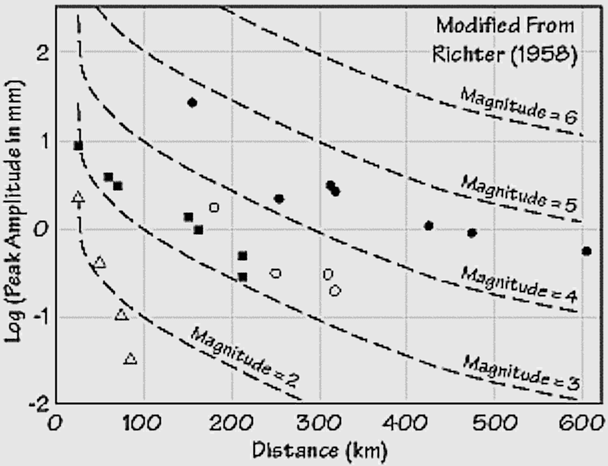
The Local Magnitude (Richter Scale)

- The local magnitude was originally defined as

$$M_L = \log_{10} A \tag{99}$$

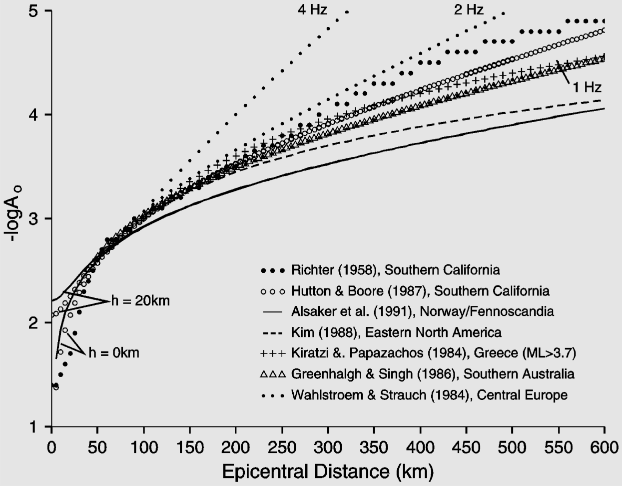
  where the maximum amplitude A of the Wood-Anderson seismometer is measured in μm at 100 km distance from the epicenter.
- $e = 1 \rightarrow$ 1 unit increase in magnitude corresponds to an increase in the instrument's amplitude by a factor 10.
- Originally only a distance correction $\sigma(\Delta) = -\log_{10} A_0(\Delta)$ for shallow earthquakes ($h \leq 15$ km) in California was provided.

Richter's Original Distance Correction



Source: C. J. Ammon, Pennsylvania State University

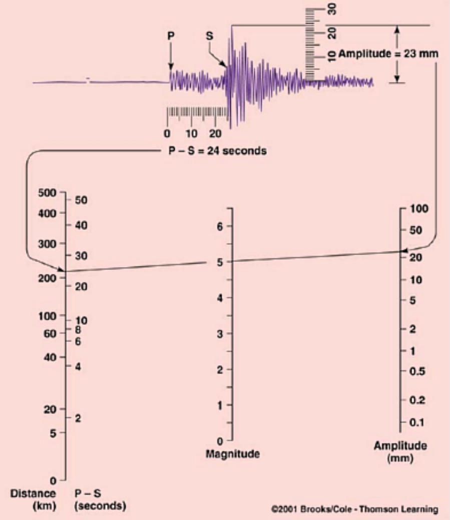
Distance Corrections for Different Regions and Depths



Source: Bormann (ed), New Manual of Seismological Observatory Practice

Intensity and Magnitude

Determining the Local Magnitude of an Earthquake



The Surface-Wave Magnitude

- Symbol: M_S or MS
- Original definition by B. Gutenberg (1945):

$$M_S = \log_{10} u_{h\,\max} + \sigma(\Delta) \tag{100}$$

where $u_{h\,\max}$ is the maximum horizontal ground displacement at periods from $T = 18\,\mathrm{s}$ to $22\,\mathrm{s}$.

- Widely used modified definition (Moscow-Prague formula, 1962):

$$M_S = \max\left\{\log_{10}\frac{|\vec{u}|}{T}\right\} + 1.66\log_{10}\Delta + 3.3 \tag{101}$$

for $2^\circ \leq \Delta \leq 160^\circ$. The maximum is taken over all periods of surface waves.

Body-Wave Magnitudes

- Two significantly different definitions
- Symbols: m_B, mB, m_b, mb,
- Original definition by B. Gutenberg (1945):

$$m_B = \max\left\{\log_{10}\frac{|\vec{u}|}{T}\right\} + \sigma(\Delta) \tag{102}$$

where $|\vec{u}|$ is analyzed for different types of body waves separately (with different functions $\sigma(\Delta)$) at periods from $T = 0.5\,\mathrm{s}$ to $12\,\mathrm{s}$.

- Alternative definition (m_b, mb) refers to higher-frequency components of P-waves only.

Scaling Properties of Earthquakes

Alternative interpretation of the seismic moment:

$$M_0 \;=\; \mu\, A\, \overline{u} \tag{103}$$

where

$$\begin{aligned} A &= \text{size of the rupture area } [\mathrm{m}^2] \\ \overline{u} &= \text{mean displacement along the rupture area } [\mathrm{m}] \end{aligned}$$

Result from the theory of crack propagation:

$$\overline{u} \;\sim\; \sqrt{A} \tag{104}$$

$$\Downarrow$$

$$\overline{u} \;\sim\; M_0^{\frac{1}{3}}, \quad A \;\sim\; M_0^{\frac{2}{3}} \tag{105}$$

Scaling Properties of Earthquakes

Rupture propagates along the rupture area at a given velocity

$$\Downarrow$$

$$\text{Duration} \quad \tau \sim \sqrt{A} \sim M_0^{\frac{1}{3}} \tag{106}$$

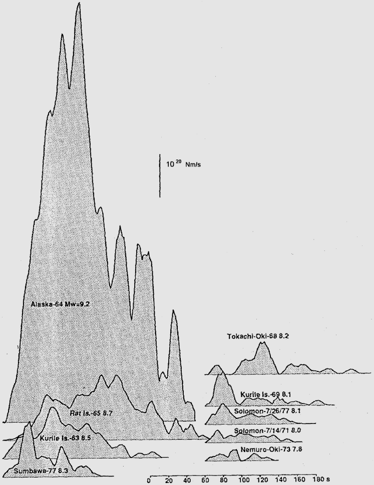
$$\Downarrow$$

$$|\vec{u}| \sim \dot{M}_0 \sim \frac{M_0}{\tau} \sim M_0^{\frac{2}{3}} \tag{107}$$

$$\Downarrow$$

Magnitude definition based on M_0 requires $e = \frac{2}{3}$.

Scaling Properties of Earthquakes



Source: Bormann (ed), New Manual of Seismological Observatory Practice

The Moment Magnitude

$$M_W = \frac{2}{3}\log_{10} M_0 - 6.1 \tag{108}$$

with M_0 in Nm

- Introduced in 1977 by H. Kanamori.
- More closely related to the strength of earthquakes at the seismic focus than to the radiated waves.

$$\Downarrow$$

Rather a tectonic than a seismological magnitude scale.

The Energy Magnitude

A crude scaling relation: particle velocity

$$|\vec{v}| \sim \frac{|\vec{u}|}{\tau} \sim \frac{M_0^{\frac{2}{3}}}{\tau} \sim M_0^{\frac{1}{3}} \tag{109}$$

$$\Downarrow$$

Total radiated kinetic energy

$$E_{\text{kin}} \sim |\vec{v}|^2\tau \sim M_0 \tag{110}$$

Potential energy equals kinetic energy in the mean.

$$\Downarrow$$

Total radiated seismic energy

$$E \sim M_0 \tag{111}$$

The Energy Magnitude

Theoretical relationship suggested by H. Kanamori (1977):

$$E \;\approx\; 5 \times 10^{-5} M_0 \tag{112}$$

Corresponding definition of the energy magnitude:

$$M_E \;=\; \tfrac{2}{3}\log_{10}\frac{E}{5 \times 10^{-5}} - 6.1 \;=\; \tfrac{2}{3}\log_{10} E - 3.2 \tag{113}$$

Up to one order of magnitude deviation from Kanamori's relationship was found for indivudual earthquakes.

$\Downarrow$

Significant differences between M_E and M_W for individual earthquakes.

Saturation of Magnitudes

All magnitude definitions based on ground displacement (M_L, M_S, ...) focus on a limited frequency / period range.

$$\Downarrow$$

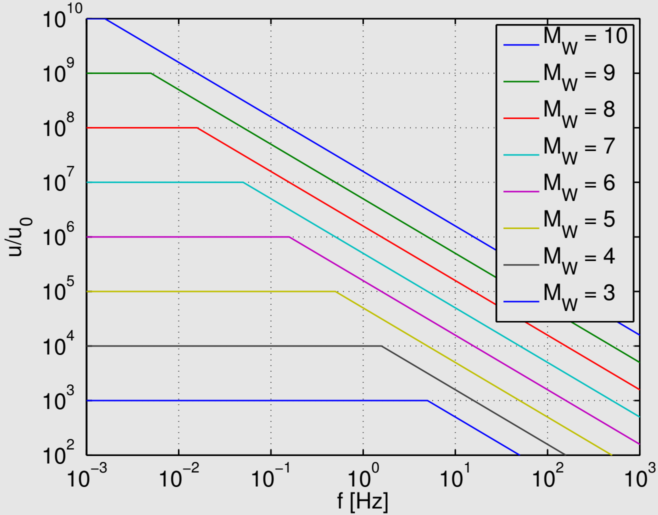
Relationship $|\vec{u}| \sim M_0^{\frac{2}{3}}$ does not always hold.

Simple model for the dependence on frequency f:

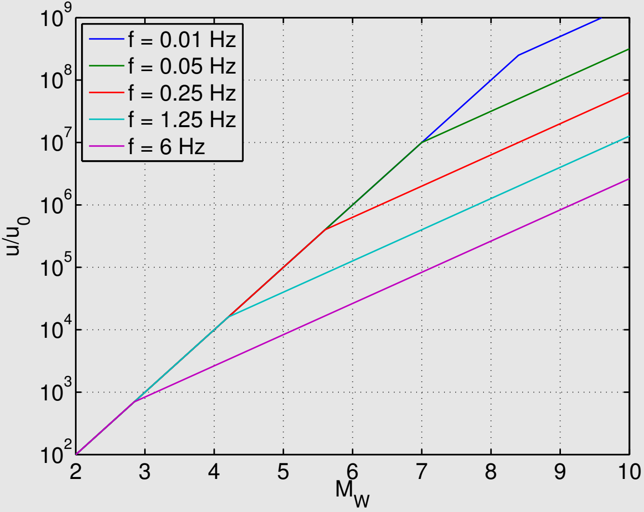
$$|\vec{u}| \sim \begin{cases} M_0^{\frac{2}{3}} & \text{for} \quad f \leq f_c \\ M_0^{\frac{2}{3}} \frac{f_c}{f} & \phantom{\text{for}} \quad f > f_c \end{cases} \tag{114}$$

with the corner frequency $f_c \sim M_0^{-\frac{1}{3}}$; $f_c \approx 0.05\,\text{Hz}$ for $M_W = 7$.

Saturation of Magnitudes



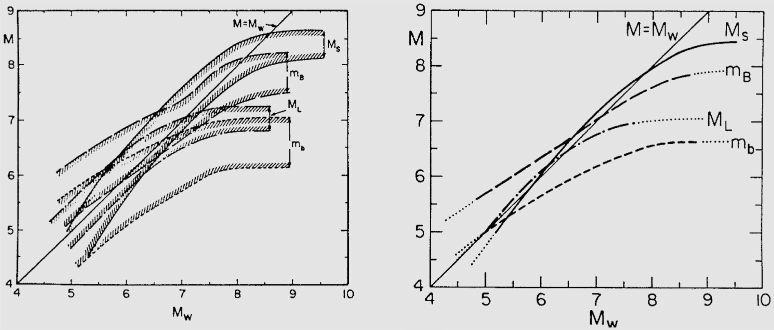
Saturation of Magnitudes



Earthquake Magnitude

Saturation of Magnitudes

- All magnitudes based on recording seismic waves fall below M_W for large earthquakes.
- Effect is stronger if short-term (high-freqency) components of the seismic waves are used.



Source: Bormann (ed), New Manual of Seismological Observatory Practice