Tsunamis

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Main Properties of Tsunamis

- Gravitiy waves in oceans with long periods between about 100 s and 10,000 s.
- Tsunamis propagate at high velocities in deep water.
- Mainly horizontal particle motion involving the entire water column down to the ocean floor.

$$\Downarrow$$

Rather small dissipation of energy.

$$\Downarrow$$

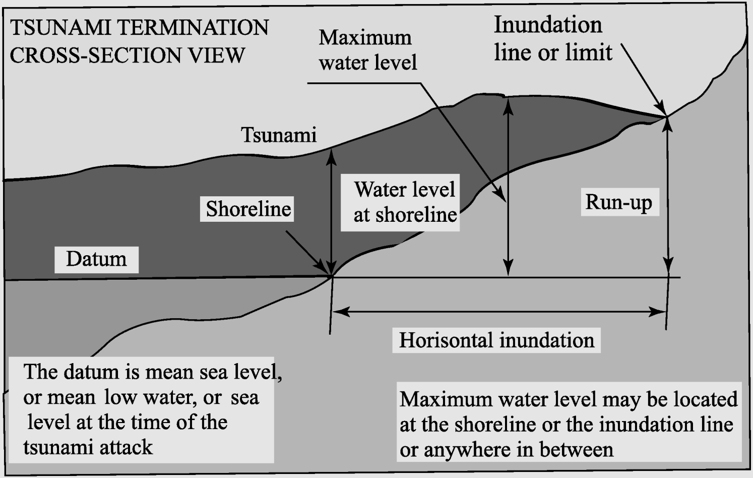
Tsunamis travel over large distances.

- Wave height increases with decreasing ocean depth.

$$\Downarrow$$

Tsunamis may reach large wave heights at the coast.

Basic Terms



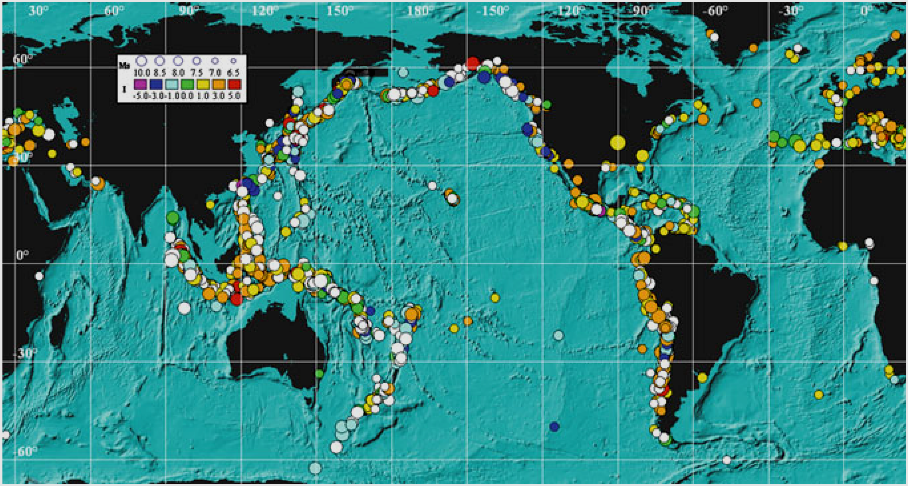
Source: Levin & Nosov, Physics of Tsunamis

Introduction

Main Sources of Tsunamis

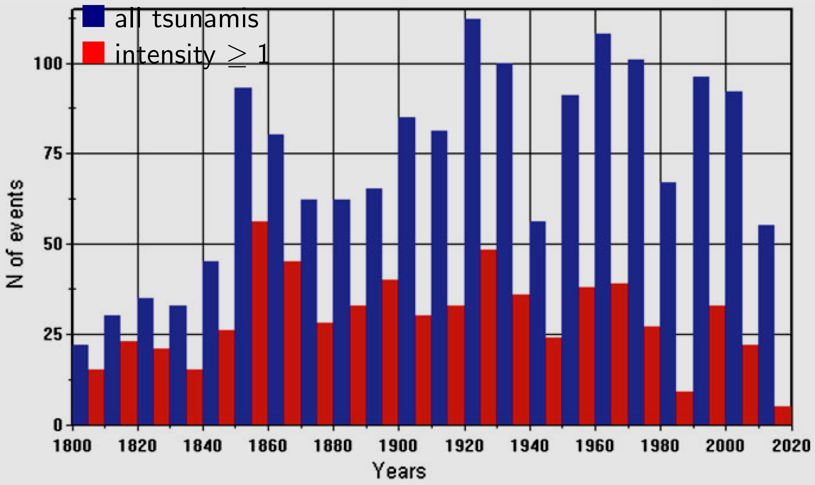
- Earthquakes (more than 90 % of all tsunamis)
- Landslides
- Volcanic eruptions
- Meteorite impact (rare)

Worldwide Distribution of Tsunami Sources from 2000 B.C. to 2014



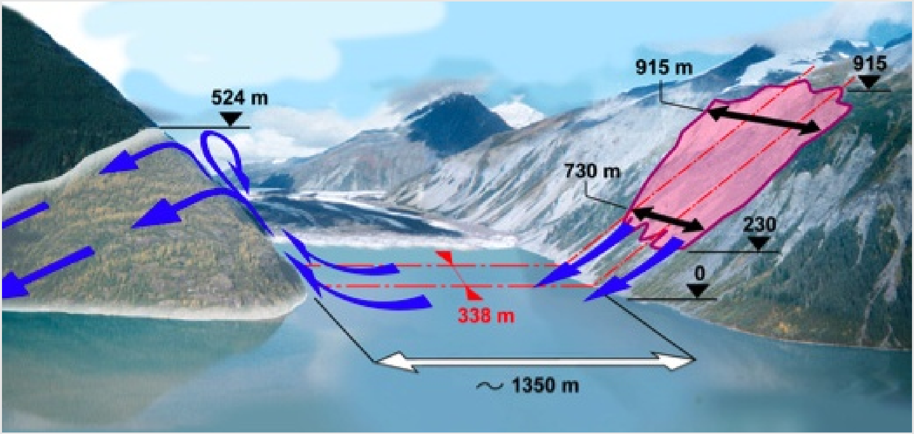
Source: Levin & Nosov, Physics of Tsunamis

Worldwide Number of Tsunamis per Decade



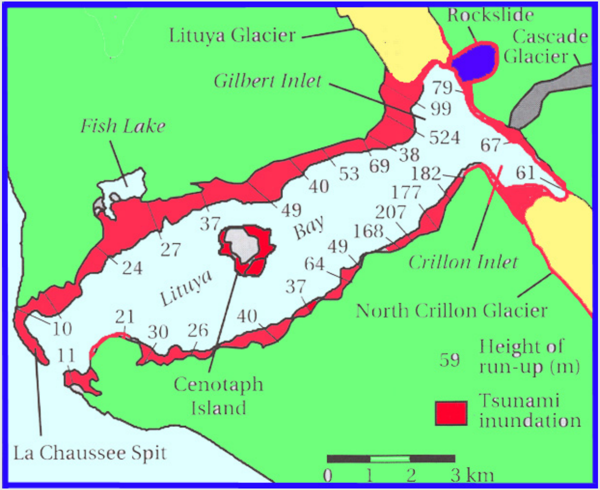
Source: Levin & Nosov, Physics of Tsunamis

The Tallest Tsunami Known so far: Lituya Bay, 1958



Source: Pararas-Carayannis, The Mega-Tsunami of July 9, 1958 in Lituya Bay, Alaska

The Tallest Tsunami Known so far: Lituya Bay, 1958



Source: Pararas-Carayannis, The Mega-Tsunami of July 9, 1958 in Lituya Bay, Alaska

The Tallest Tsunamis 2000–2014

Date	Location	M_W	H_{max} [m]	Death toll
11.03.2011	Japan	9.0	56	18,482
24.12.2004	Indonesia, Sumatra	9.1	51	227,899
27.02.2010	Chile	8.8	29	156
29.09.2009	Samoa	8.1	22	192
15.11.2006	Russia, Kuril Islands	8.3	22	0
17.07.2006	Indonesia, South of Java	7.7	21	802
25.10.2010	Indonesia, Sumatra	7.8	17	431

Intensity and Magnitude

Types of Intensity and Magnitude Scales

Three different types of scales:

- Intensity scales characterizing the effect of a tsunami on humans and their structures (Sieberg-Ambraseys scale, Papadopoulos-Imamura scale).
- Intensity scales based on measurements of wave height at the coast (Imamura-Iida scale, Soloviev-Imamura scale).
- Magnitude scales characterizing the stength of a tsunami inpendent of distance between source and coast and the shape of the coast (Abe-Hatori scale, Murty-Loomis scale).

The Sieberg-Ambraseys Scale

- Originally introduced by A. H. Sieberg (1927), modified by N. N. Ambraseys (1962).
- Six-point scale from 1 = very light to 6 = disastrous.

The Papadopoulos-Imamura Scale

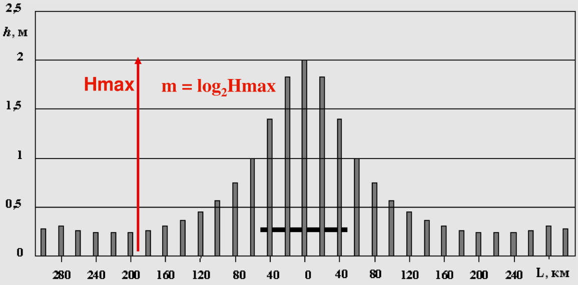
- Introduced by G. A. Papadopoulos and F. Imamura (2001).
- 12-point scale in analogy to the Mercalli scale for earthquakes from I = not felt to XII = destructive.

The Imamura-Iida Scale

- Introduced by A. Imamura (1942), modified by K. Iida (1956).
- Defined as

$$m = \log_2 H_{\max} \tag{1}$$

where $H_{\max}$ is the maximum wave height.



Source: Gusiakov, Tsunami Quantification

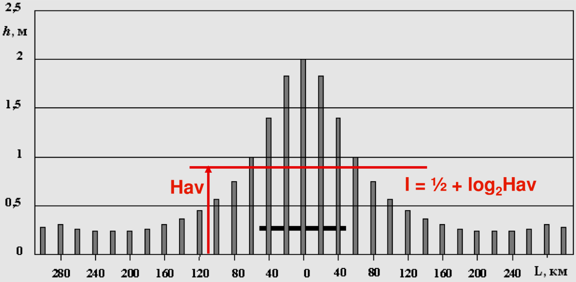
- Originally termed magnitude.

The Soloviev-Imamura Scale

- Modification of the Imamura-Iida scale by S. Soloviev (1972).
- Defined as

$$I = \frac{1}{2} + \log_2 H_{av} \quad (2)$$

where H_{av} is the average wave height along the nearest coast.



Source: Gusiakov, Tsunami Quantification

- Widely used in many tsunami catalogs.

The Abe-Hatori Scale

- Introduced in 1979 by K. Abe.
- First attempt ro define a tsunami magnitude taking into account the distance from the source:

$$M_t = a \log_{10} H_{\max} + b \log_{10} \Delta + D \quad (3)$$

where

$$\begin{aligned} H_{\max} &= \text{maximum wave amplitude at the coast} \\ \Delta &= \text{distance} \\ a, b, D &= \text{constants} \end{aligned}$$

The Murty-Loomis Scale

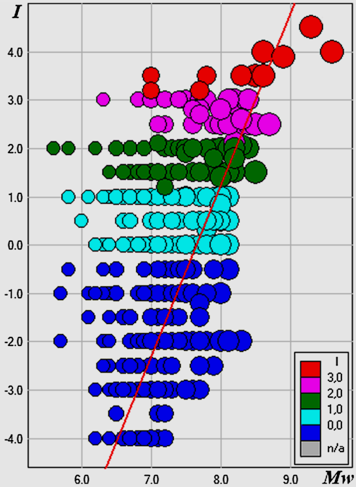
- Introduced in 1980 by T. S. Murty and H. G. Loomis.
- Based on the total potential energy E (in J here, originally in ergs):

$$ML = 2\left(\log_{10} E - 12\right) \qquad (4)$$

- Well- defined and
- theoretically a good measure of the strength of a tsunami,
- but suffers from the problem of determining the total potential energy.

Intensity and Magnitude

Tsunami Intensity (Soloviev-Imamura) vs. Earthquake Magnitude



Source: Gusiakov, Pure Appl. Geophys, 2015

Starting Point

Cauchy equations for the displacement $\vec{u}(\vec{x}, t)$ including gravity:

$$\rho \frac{\partial^2}{\partial t^2}\vec{u} \;=\; \mathrm{div}(\boldsymbol{\sigma}) - \rho g \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{5}$$

Incompressible and inviscid fluid instead of an elastic medium:

- Volumetric strain for small deformation

$$\epsilon_v \;=\; \mathrm{trace}(\boldsymbol{\epsilon}) \;=\; \mathrm{div}(\vec{u}) \;=\; 0 \tag{6}$$

- Stress tensor

$$\boldsymbol{\sigma} \;=\; -p\mathbf{1} \tag{7}$$

with the fluid pressure $p(\vec{x}, t)$

Starting Point

↓

$$\rho \frac{\partial^2}{\partial t^2}\vec{u} \;=\; -\nabla p - \rho g \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \;=\; -\nabla\left(p + \rho g x_3\right) \tag{8}$$

Harmonic Plane Wave

$$\vec{u}(\vec{x}, t) \;=\; e^{i\omega(t - \vec{s}\cdot\vec{x})}\vec{a} \;=\; e^{i\left(\omega t - \vec{k}\cdot\vec{x}\right)}\vec{a} \tag{9}$$

where

ω = angular frequency [s^{-1}]

$\vec{s}$ = slowness vector [$\frac{s}{m}$]

$\vec{k}$ = $\omega\vec{s}$ = wave number vector [m^{-1}]

$\vec{a}$ = amplitude vector [m]

Non-Existence of S-Waves

$$\rho \frac{\partial^2}{\partial t^2}\vec{u} \;=\; -\rho\omega^2 e^{i\left(\omega t - \vec{k}\cdot\vec{x}\right)}\vec{a} \;=\; -\nabla\left(p + \rho g x_3\right) \tag{10}$$

↓

$$\operatorname{curl}\left(\rho \frac{\partial^2}{\partial t^2}\vec{u}\right) \;=\; -\rho\omega^2 e^{i\left(\omega t - \vec{k}\cdot\vec{x}\right)}\left(-i\vec{k}\right)\times\vec{a} \tag{11}$$

$$=\; -\operatorname{curl}\left(\nabla\left(p + \rho g x_3\right)\right) \;=\; 0 \tag{12}$$

↓

$$\vec{k}\times\vec{a} \;=\; \vec{0} \tag{13}$$

↓

$\vec{k}$ and $\vec{a}$ must be parallel; only P-waves are possible.

Solution for a Harmonic Plane Wave

$$\mathrm{div}(\vec{u}) \;=\; e^{i\left(\omega t - \vec{k}\cdot\vec{x}\right)}\left(-i\vec{k}\right)\cdot\vec{a} \;=\; 0 \tag{14}$$

$$\Downarrow$$

$$k_1^2 + k_2^2 + k_3^2 \;=\; 0 \tag{15}$$

For a wave propagating in x_1 direction:

$$\vec{k} \;=\; \begin{pmatrix} k \\ 0 \\ \pm ik \end{pmatrix} \quad \text{and} \quad \vec{a} \;=\; \begin{pmatrix} a \\ 0 \\ \pm ia \end{pmatrix} \tag{16}$$

$$\Downarrow$$

$$\vec{u}(\vec{x},t) \;=\; e^{i(\omega t - kx_1 \mp ikx_3)}\vec{a} \;=\; a\, e^{\pm kx_3}\, e^{i(\omega t - kx_1)}\begin{pmatrix} 1 \\ 0 \\ \pm i \end{pmatrix} \tag{17}$$

Solution for a Harmonic Plane Wave

Consider domain $x_3 \leq 0$ $\rightarrow$ Only solution with "+" makes sense.

$$\vec{u}(\vec{x}, t) \;=\; a\, e^{kx_3} e^{i(\omega t - kx_1)} \begin{pmatrix} 1 \\ 0 \\ i \end{pmatrix} \tag{18}$$

- Prograde particle movement on circular orbits
- Depth of penetration

$$d \;=\; \frac{1}{k} \;=\; \frac{L}{2\pi} \tag{19}$$

with the wavelength $L = \frac{2\pi}{k}$

Velocity of Propagation

For solving Eq. 8 write $\rho\frac{\partial^2}{\partial t^2}\vec{u}$ as

$$\rho\frac{\partial^2}{\partial t^2}\vec{u} = -\rho\omega^2 e^{i\left(\omega t-\vec{k}\cdot\vec{x}\right)}\vec{a} = -\frac{\rho\omega^2 a}{k}e^{i\left(\omega t-\vec{k}\cdot\vec{x}\right)}\vec{k} \quad (20)$$

$$= \nabla\left(-\frac{i\rho\omega^2 a}{k}e^{i\left(\omega t-\vec{k}\cdot\vec{x}\right)}\right) \quad (21)$$

⬇

$$\nabla\left(-\frac{i\rho\omega^2 a}{k}e^{i\left(\omega t-\vec{k}\cdot\vec{x}\right)} + p + \rho g x_3\right) = \vec{0} \quad (22)$$

⬇

$$-\frac{i\rho\omega^2 a}{k}e^{i\left(\omega t-\vec{k}\cdot\vec{x}\right)} + p + \rho g x_3 = \text{const} \quad (23)$$

Velocity of Propagation

Free ocean surface with $p = \text{const}$ at $x_3 = u_3(x_1, x_2, 0)$

$$\Downarrow$$

$$-\frac{i\rho\omega^2 a}{k} e^{i(\omega t - kx_1)} + \rho g i a\, e^{i(\omega t - kx_1)} = \text{const} \tag{24}$$

$$\Downarrow$$

$$\omega^2 = gk \tag{25}$$

Velocity of Propagation

Phase Velocity:

$$v_{\mathrm{ph}} = \frac{\omega}{k} = \sqrt{\frac{g}{k}} = \sqrt{\frac{gL}{2\pi}} \tag{26}$$

$\Downarrow$

Strong dispersion

Group Velocity:

$$v_{\mathrm{gr}} = \frac{d\omega}{dk} = \frac{1}{2}\sqrt{\frac{g}{k}} = \frac{1}{2} v_{\mathrm{ph}} \tag{27}$$

Boundary Condition at the Ocean Floor

Consider domain $-H \leq x_3 \leq 0$ with a given ocean depth H.

$\Downarrow$

Solution must meet the condition $u_3(x_1, x_2, -H, t) = 0$.

$\Downarrow$

Versions with $+$ and $-$ in Eq. 17 must be superposed:

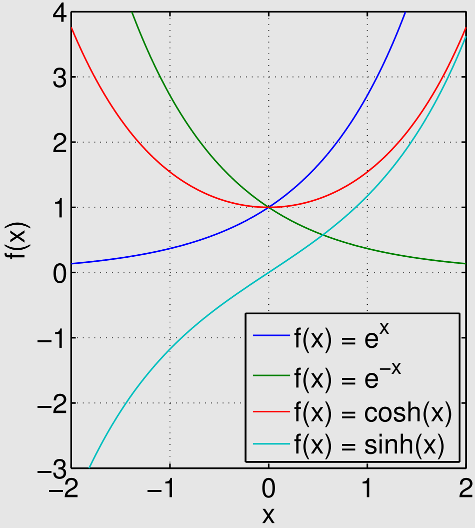
$$\vec{u}(\vec{x}, t) = a_+ \, e^{kx_3} e^{i(\omega t - kx_1)} \begin{pmatrix} 1 \\ 0 \\ i \end{pmatrix} + a_- \, e^{-kx_3} e^{i(\omega t - kx_1)} \begin{pmatrix} 1 \\ 0 \\ -i \end{pmatrix} \quad (28)$$

$$= \frac{a \, e^{i(\omega t - kx_1)}}{e^{kH} + e^{-kH}} \left(e^{k(x_3+H)} \begin{pmatrix} 1 \\ 0 \\ i \end{pmatrix} + e^{-k(x_3+H)} \begin{pmatrix} 1 \\ 0 \\ -i \end{pmatrix} \right) \quad (29)$$

The Hyperbolic Cosine and Sine Functions

$$\cosh(x) = \frac{e^x + e^{-x}}{2} \quad (30)$$

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \quad (31)$$



Ocean Waves at Finite Ocean Depth

Particle displacement of a Harmonic Plane Wave

Particle displacement expressed in terms of $\cosh(x)$ and $\sinh(x)$:

$$\vec{u}(\vec{x}, t) = \frac{a\, e^{i(\omega t - k x_1)}}{\cosh(kH)} \begin{pmatrix} \cosh(k(x_3 + H)) \\ 0 \\ i \sinh(k(x_3 + H)) \end{pmatrix} \tag{32}$$

- Prograde particle movement on elliptical orbits.
- Orbits are always wider than high; height-to-width ratio:

$$S = \frac{\sinh(k(x_3 + H))}{\cosh(k(x_3 + H))} = \tanh(k(x_3 + H)) \tag{33}$$

- Horizontal amplitude at the surface $= a$.

Particle displacement of a Harmonic Plane Wave

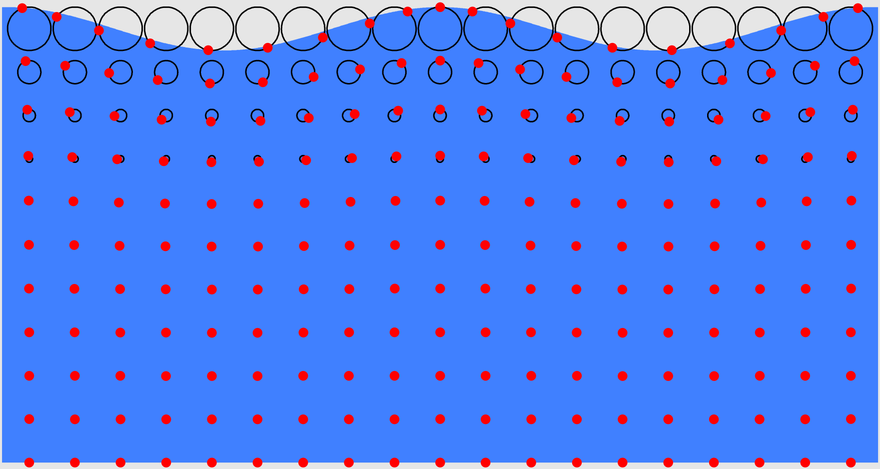
- Wave height (vertical amplitude at the surface):

$$h = a \tanh(kH) \tag{34}$$

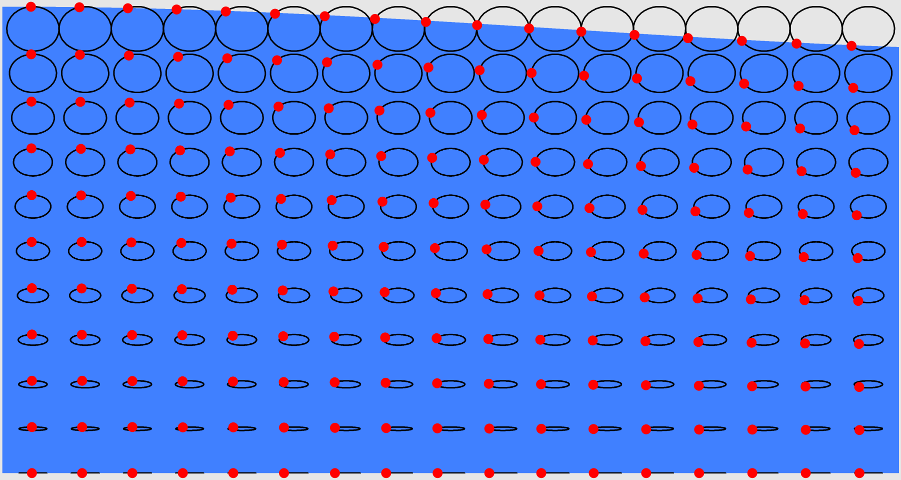
- Orbits are almost circular ($S \to 1$) for

$$k(x_3 + H) \to \infty \iff x_3 + H \gg L$$

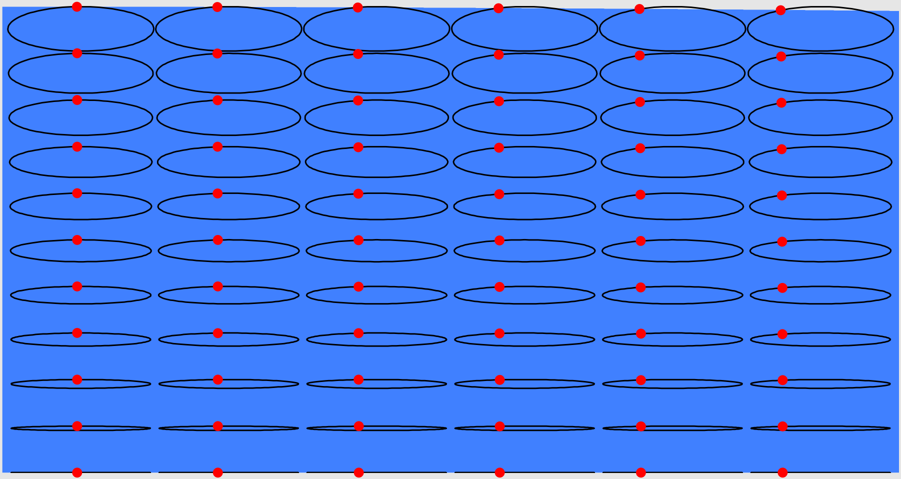
Particle Orbits for $L/H = 1$



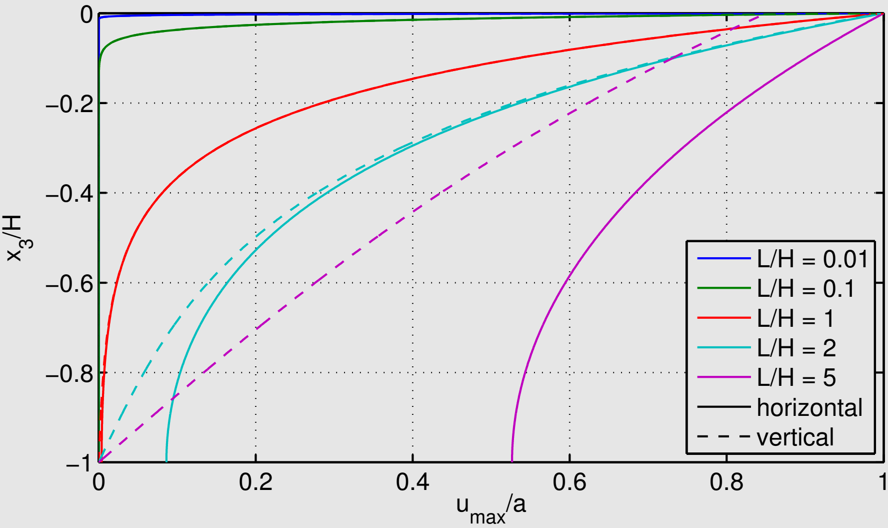
Particle Orbits for $L/H = 5$



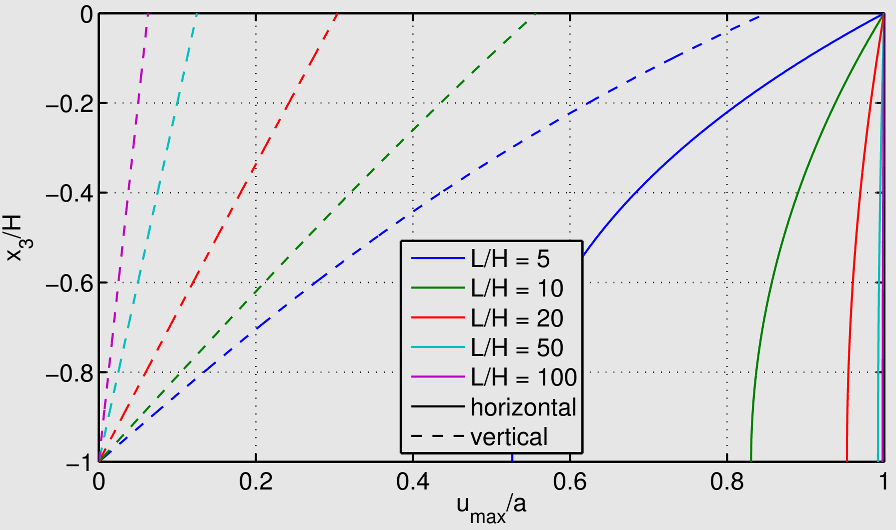
Particle Orbits for $L/H = 20$



Maximum Particle Displacment of a Harmonic Plane Wave



Maximum Particle Displacment of a Harmonic Plane Wave



Velocity of Propagation

$$\rho \frac{\partial^2}{\partial t^2}\vec{u} = -\frac{\rho\omega^2 a\, e^{i(\omega t - kx_1)}}{\cosh(kH)} \begin{pmatrix} \cosh(k(x_3+H)) \\ 0 \\ i\sinh(k(x_3+H)) \end{pmatrix} \tag{35}$$

$$= \nabla\left(-\frac{i\rho\omega^2 a}{k\cosh(kH)} e^{i(\omega t - kx_1)} \cosh(k(x_3+H))\right) \tag{36}$$

$$\Downarrow$$

$$\nabla\left(-\frac{i\rho\omega^2 a}{k\cosh(kH)} e^{i(\omega t - kx_1)} \cosh(k(x_3+H)) + p + \rho g x_3\right) = \vec{0} \tag{37}$$

$$\Downarrow$$

$$-\frac{i\rho\omega^2 a}{k\cosh(kH)} e^{i(\omega t - kx_1)} \cosh(k(x_3+H)) + p + \rho g x_3 = \text{const} \tag{38}$$

Velocity of Propagation

Free ocean surface with $p = \text{const}$ at $x_3 = u_3(x_1, x_2, 0)$

$$\Downarrow$$

$$-\frac{i\rho\omega^2 a}{k} e^{i(\omega t - kx_1)} + \rho g \frac{a\, e^{i(\omega t - kx_1)}}{\cosh(kH)} i \sinh(kH) \;=\; \text{const} \tag{39}$$

$$\Downarrow$$

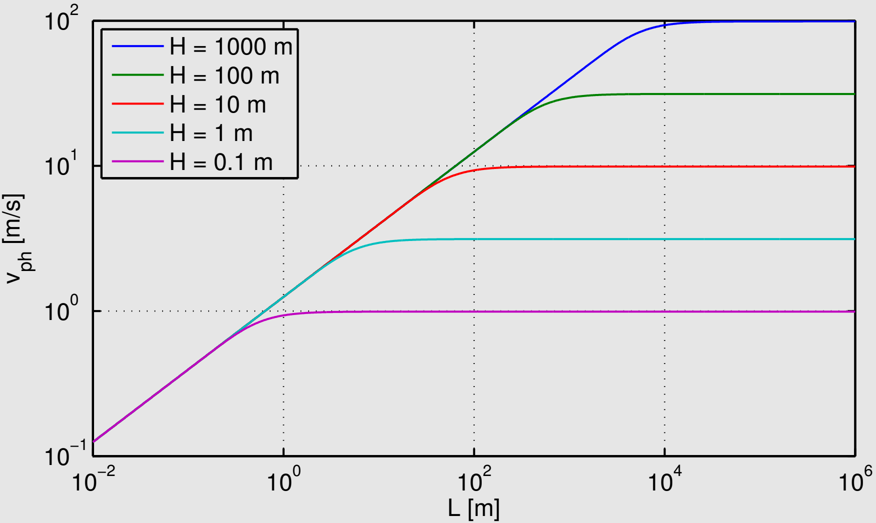
$$\omega^2 \;=\; gk \frac{\sinh(kH)}{\cosh(kH)} \;=\; gk\,\tanh(kH) \tag{40}$$

Phase Velocity:

$$v_{\text{ph}} \;=\; \frac{\omega}{k} \;=\; \sqrt{\frac{g\,\tanh(kH)}{k}} \tag{41}$$

Ocean Waves at Finite Ocean Depth

Velocity of Propagation



Velocity of Propagation

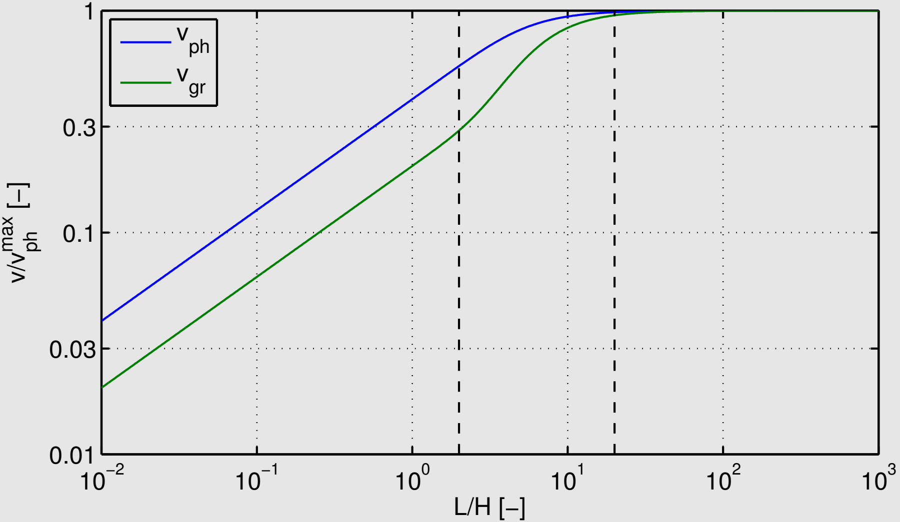
Maximum phase velocity (at long wavelengths, $kH \to 0$):

$$v_{\text{ph}}^{\text{max}} = \sqrt{gH} \tag{42}$$

Group Velocity:

$$\begin{aligned} v_{\text{gr}} &= \frac{d\omega}{dk} = \frac{g}{2\omega}\left(\tanh(kH) + \frac{kH}{\cosh^2(kH)}\right) \\ &= \frac{1}{2} v_{\text{ph}} \left(1 + \frac{kH}{\sinh(kH)\,\cosh(kH)}\right) \end{aligned}$$

Velocity of Propagation



Ocean Waves at Finite Ocean Depth

Regimes of Ocean Wave Propagation

Deep water regime: $L/H \leq 2$

- Particles move on almost circular orbits.
- Particle movement is practically limited to a depth less than one wavelength.
- Phase velocity and group velocity depend on the wavelength, but not on ocean depth:

$$v_{\rm ph} \approx \sqrt{\frac{g}{k}} = \sqrt{\frac{gL}{2\pi}}, \quad v_{\rm gr} \approx \frac{1}{2} v_{\rm ph}$$

- Strong dispersion

Regimes of Ocean Wave Propagation

Shallow water regime: $L/H \geq 20$

- Particles move on elliptical orbits.
- Horizontal particle movement persists down to the ocean floor.
- Phase velocity and group velocity depend only on ocean depth:

$$v_{\text{ph}} \approx v_{\text{gr}} \approx \sqrt{gH}$$

- No dispersion

Dispersion

Examples of tsunami wave dispersion in a 4000 m deep ocean (symmetric propagation to the left and to the right):

- bell-shaped (Gaussian) wave
- boxcar-shaped wave
- double boxcar-shaped wave
- step-like wave

The Fluid Pressure

From Eqs. 38 and 40 with $p = 0$ at the ocean surface:

$$p(\vec{x}, t) = -\rho g x_3 + \frac{i\rho\omega^2 a}{k\cosh(kH)} e^{i(\omega t - kx_1)} \cosh(k(x_3 + H)) \tag{43}$$

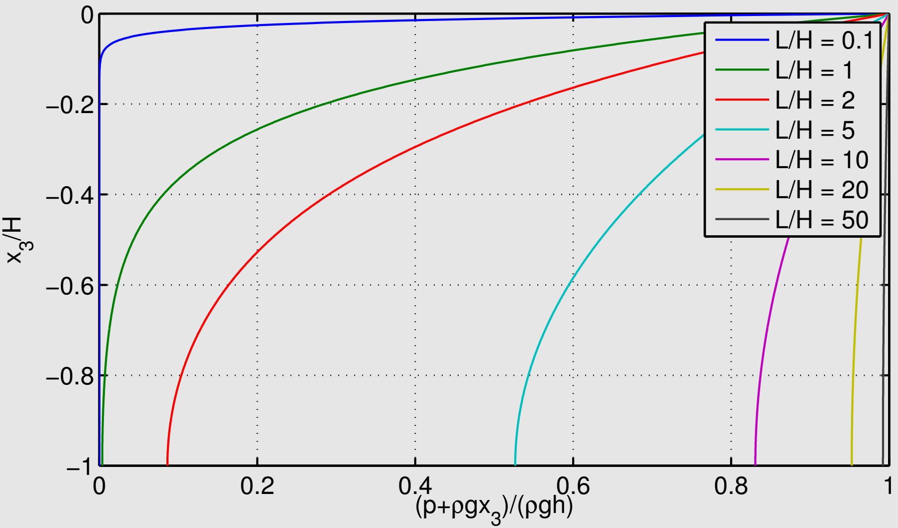
$$= -\rho g x_3 + \frac{i\rho g\ \tanh(kH) a}{\cosh(kH)} e^{i(\omega t - kx_1)} \cosh(k(x_3 + H)) \tag{44}$$

$$= -\rho g x_3 + \rho g h(x_1, x_2, t) \frac{\cosh(k(x_3 + H))}{\cosh(kH)} \tag{45}$$

with the wave height

$$h(x_1, x_2, t) = u_3(x_1, x_2, 0, t) = a\, e^{i(\omega t - kx_1)} i \tanh(kH) \tag{46}$$

The Fluid Pressure



Ocean Waves at Finite Ocean Depth

The Fluid Pressure

Pressure variation at the ocean floor:

$$p(x_1, x_2, -H, t) - \rho g H = \frac{\rho g h(x_1, x_2, t)}{\cosh(kH)} \qquad (47)$$

Deep water regime $(L/H \leq 2)$: $< 10\,\%$ of the near-surface variation

Shallow water regime $(L/H \geq 20)$: $> 95\,\%$ of the near-surface variation

$$\Downarrow$$

Most important component of tsunami warning systems beyond earthquake registration.

The Shallow Water Approximation

Allows for generalization to

- non-harmonic waves and
- non-constant ocean depth (refraction, amplification)

in the limit $kH \to 0$.

Approximations:

- Vertical particle displacement is negligible and is only used for maintaining the incompressibility.
- Horizontal particle displacement depends on x_1 and x_2 only.
- Hydrostatic vertical pressure profile.
- Consider all vectors as a two-component vectors from now on.

The Shallow Water Approximation

Mass balance:

$$\rho \frac{\partial}{\partial t} h(\vec{x}, t) = -\text{div}\left(\int_{-H}^{0} \rho \frac{\partial}{\partial t} \vec{u}(\vec{x}, t)\, dx_3 \right) \tag{48}$$

$$= -\text{div}\left(H\rho \frac{\partial}{\partial t} \vec{u}(\vec{x}, t) \right) \tag{49}$$

The Shallow Water Approximation

Cauchy equations for an inviscid fluid (horizontal component only):

$$\rho \frac{\partial^2}{\partial t^2} \vec{u}(\vec{x}, t) \;=\; -\nabla p(\vec{x}, t) \tag{50}$$

Hydrostatic pressure distribution with free surface:

$$p(\vec{x}, x_3, t) \;=\; -\rho g x_3 + \rho g\, h(\vec{x}, t) \tag{51}$$

$$\Downarrow$$

$$\frac{\partial^2}{\partial t^2} \vec{u}(\vec{x}, t) \;=\; -g \nabla h(\vec{x}, t) \tag{52}$$

The Shallow Water Approximation

Insert Eq. 52 into the derivative of Eq. 49:

$$\frac{\partial^2}{\partial t^2} h(\vec{x}, t) = -\mathrm{div}\left(H \frac{\partial^2}{\partial t^2} u(\vec{x}, t)\right) \tag{53}$$

$$= -\mathrm{div}\left(-gH \nabla h(\vec{x}, t)\right) \tag{54}$$

$$= \mathrm{div}\left(v^2 \nabla h(\vec{x}, t)\right) \tag{55}$$

is a wave equation with the velocity

$$v = \sqrt{gH} \tag{56}$$

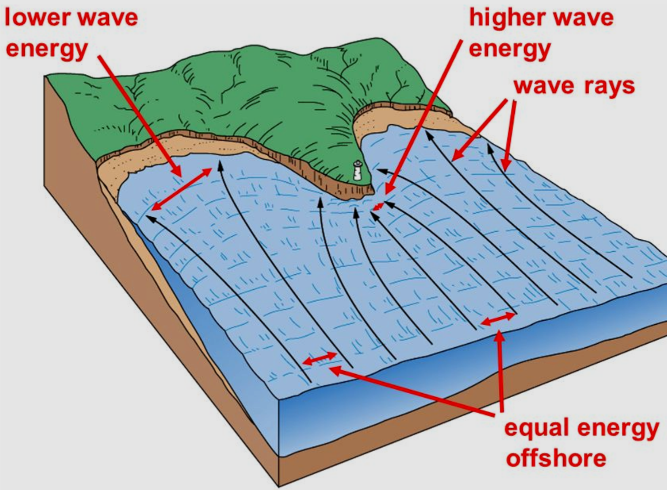
already known for the shallow water regime.

Wave Propagation at Non-Constant Water Depth

- Reflection and refraction according to Snell's law, similar to seismic waves.
- Only significant in the shallow water regime.
- Small gradual changes in water depth: continuous refraction towards smaller water depth without significant reflection.
- Amplitude?

Shallow Water Waves

Wave Shoaling



Source: Carpenter, Ocean Waves

Wave Shoaling

Modify the approach considered in assignment 3:

$$h(\vec{x}, t) \; = \; f\left(t - \psi(\vec{x})\right) a(\vec{x}) \tag{57}$$

Wave equation:

$$\begin{aligned} \frac{\partial^2}{\partial t^2} h \;&=\; f'' a \;=\; \operatorname{div}\left(v^2 \nabla h\right) \;=\; \operatorname{div}\left(v^2 \nabla (fa)\right) \\ &=\; f'' v^2 |\nabla \psi|^2 a - f' \left(\operatorname{div}\left(v^2 a \nabla \psi\right) + v^2 \nabla a \cdot \nabla \psi\right) \\ &\quad +f \operatorname{div}\left(v^2 \nabla a\right) \end{aligned} \tag{58}$$

$$\Downarrow$$

$$v^2 |\nabla \psi|^2 \;=\; 1 \tag{59}$$

$$\left(\operatorname{div}\left(v^2 a \nabla \psi\right) + v^2 \nabla a \cdot \nabla \psi\right) \;=\; 0 \tag{60}$$

$$\operatorname{div}\left(v^2 \nabla a\right) \;=\; 0 \tag{61}$$

Wave Shoaling

Define

$$\vec{q} \;=\; a^2 v^2 \nabla\psi \;=\; a^2 \vec{v} \tag{62}$$

where $\vec{v} = v^2\nabla\psi$ is the velocity vector in direction of wave propagation.

$$\Downarrow$$

$$\begin{aligned} \mathrm{div}(\vec{q}) &= a\,\mathrm{div}\left(av^2\nabla\psi\right) + \nabla a \cdot \left(av^2\nabla\psi\right) && (63)\\ &= a\left(\mathrm{div}\left(av^2\nabla\psi\right) + \nabla a \cdot v^2\nabla\psi\right) && (64)\\ &= 0 && (65) \end{aligned}$$

according to Eq. 60.