Geothermics and Geothermal Energy
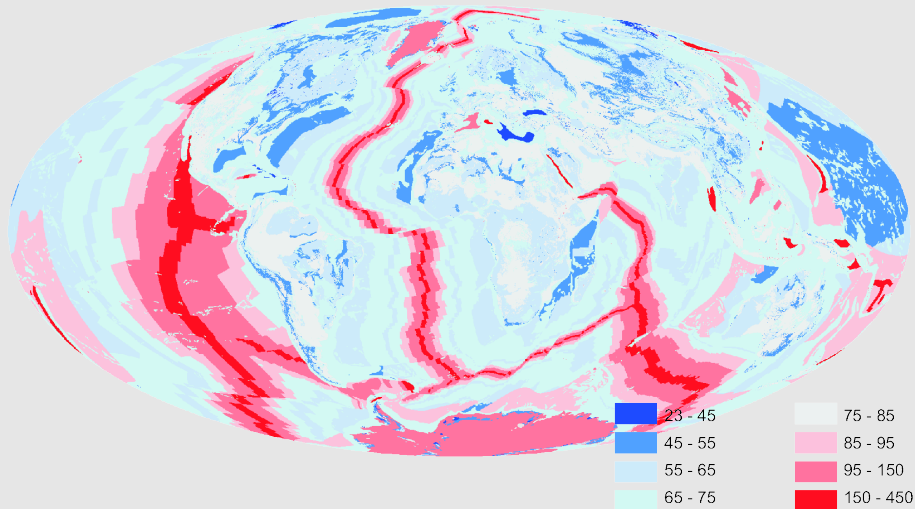# The Temperature in Earth's Crust

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Earth's Surface Heat Flux

Regional Variation of the Surface Heat Flux Density

23 - 45		75 - 85	
45 - 55		85 - 95	
55 - 65		95 - 150	
65 - 75		150 - 450	

Source: Davies & Davies, Earth's surface heat flux, Solid Earth, 2010

Earth's Surface Heat Flux

Definition of Heat Flux Density

$\vec{q}$ = energy per time and cross section area $[\frac{\text{W}}{\text{m}^2}]$

Earth's Mean Surface Heat Flux Density

Continental crust (40 % of total surface): $q = 70.9 \frac{\text{mW}}{\text{m}^2}$
Oceanic crust (60 % of total surface): $q = 105.4 \frac{\text{mW}}{\text{m}^2}$
Overall mean: $q = 91.6 \frac{\text{mW}}{\text{m}^2}$

- q is more than 4 orders of magnitude smaller than the solar constant $S = 1367 \frac{\text{W}}{\text{m}^2}$.
- q describes the long-term mean energy balance ot the solid Earth.
- q reflects the ongoing cooling of Earth and from radiogenic heat production in the upper continental crust.

The Temperature in Earth's Crust

A Typical Continental Geotherm

Fundamentals – Heat Transport
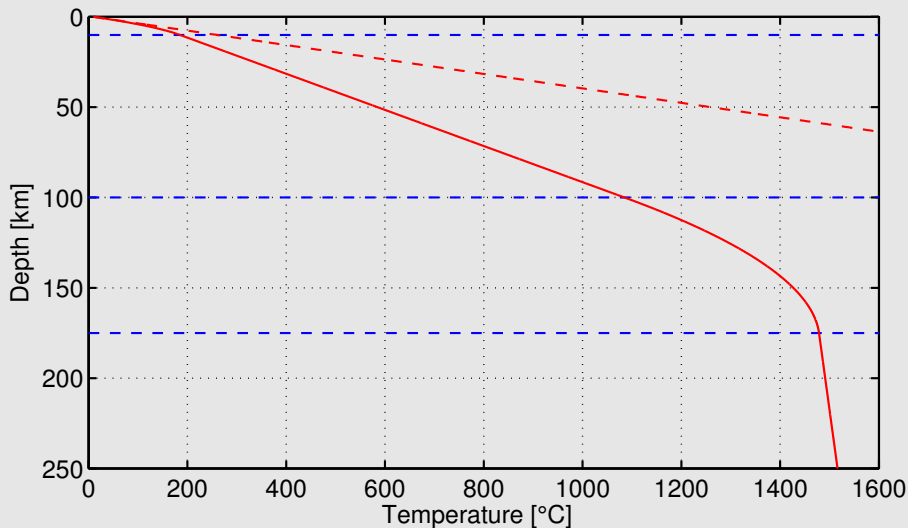
Fourier's Law of Heat Conduction (1822)

1. Heat flow follows the direction of steepest descent of the temperature field $T(\vec{x}, t) = T(x, y, z, t)$.

2. Heat flow is proportional to the decrease of temperature per length:

$$\vec{q}(\vec{x}, t) \; = \; -\lambda \nabla T(\vec{x}, t) \; = \; -\lambda \begin{pmatrix} \frac{\partial}{\partial x} T(x, y, z, t) \\ \frac{\partial}{\partial y} T(x, y, z, t) \\ \frac{\partial}{\partial z} T(x, y, z, t) \end{pmatrix} \tag{1}$$

with

$$\begin{aligned} \vec{q}(\vec{x}, t) \; &= \; \text{heat flux density (energy per area and time) } [\tfrac{W}{m^2}] \\ \lambda \; &= \; \text{thermal conductivity } [\tfrac{W}{m\,K}] \end{aligned}$$

Fundamentals – Heat Transport

The Thermal Conductivity

Typical Values:

Material	$\lambda \left[\frac{W}{m\,K}\right]$
diamond	2300
iron	80
sand	0.6
polyethylene (PE)	0.48
expanded polystyrene (EPS)	0.033
water	0.6
air	0.026

Rocks	$\lambda \left[\frac{W}{m\,K}\right]$
granite	2.8
basalt	2
dolomite	2.5
limestone	2.5
sandstone	2.5
clay	1.4
widely used value	2.5

Fundamentals – Heat Transport

The Equation of Continuity (Energy Balance)

rate of change of the thermal energy contained in a given volume

=

energy per time entering at the boundaries
-
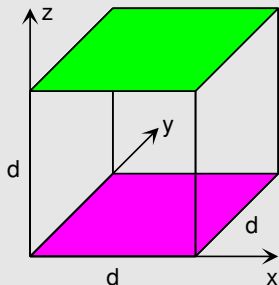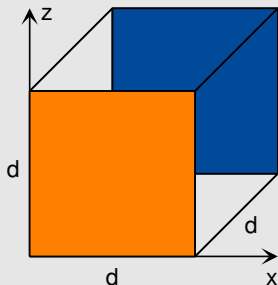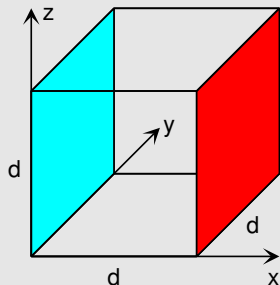energy per time leaving at the boundaries

+

energy per time generated within the volume
-
energy per time consumed within the volume

Fundamentals – Heat Transport

The Equation of Continuity (Energy Balance)

Energy balance of a cube without heat production:

Change in thermal energy E contained in the cube:

$$\frac{\partial E}{\partial t} \;=\; q_x d^2 - q_x d^2 + q_y d^2 - q_y d^2 + q_z d^2 - q_z d^2 \qquad (2)$$

The Equation of Continuity (Energy Balance)

Change in energy density e (thermal energy per volume):

$$
\begin{aligned}
\frac{\partial e}{\partial t} &= \frac{\frac{\partial E}{\partial t}}{d^3} \\
&= \frac{q_x - q_x}{d} + \frac{q_y - q_y}{d} + \frac{q_z - q_z}{d} \\
&\rightarrow -\frac{\partial q_x}{\partial x} - \frac{\partial q_y}{\partial y} - \frac{\partial q_z}{\partial z} \quad \text{for} \quad d \rightarrow 0 \\
&= -\text{div}\,(\vec{q}) \tag{3}
\end{aligned}
$$

with the divergence operator

$$
\text{div}\,(\vec{q}) = \frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z} \tag{4}
$$

The Specific Heat Capacity

Volumetric heat capacity

$$c_{\mathsf{vol}} = \frac{\partial e}{\partial T} \tag{5}$$

describes the change in thermal energy density e $[\frac{\mathsf{J}}{\mathsf{m}^3}]$ with temperature T.

Unit: $\frac{\mathsf{J}}{\mathsf{m}^3\,\mathsf{K}}$

Specific heat capacity

$$c = \frac{c_{\mathsf{vol}}}{\rho} = \frac{1}{\rho}\frac{\partial e}{\partial T} \tag{6}$$

is measured per mass instead of per volume (ρ = density).

Unit: $\frac{\mathsf{J}}{\mathsf{kg}\,\mathsf{K}}$

The Specific Heat Capacity

Molar heat capacity

$$c_{mol} \;=\; Mc \;=\; \frac{M}{\rho} \frac{\partial e}{\partial T} \tag{7}$$

is measured per mol instead of per kg (M = molar mass).

Unit: $\frac{J}{mol\,K}$

Dulong-Petit law:

$$c_{mol} \;\approx\; 3R \tag{8}$$

with the gas constant $R = 8.314 \frac{J}{mol\,K}$ for most crystalline solids.

Fundamentals – Heat Transport

The Specific Heat Capacity

Typical values at standard conditions:

Material	c $[\frac{J}{kg\,K}]$
diamond	509
iron	450
sand	550
polyethylene (PE)	1250
expanded polystyrene (EPS)	1500
water	4187
air	1005

Rocks	c $[\frac{J}{kg\,K}]$
granite	1000
basalt	850
dolomite	1000
limestone	900
sandstone	900
clay	1100
widely used value	900

Fundamentals – Heat Transport

The Heat Conduction Equation (Energy Balance + Fourier's Law)

General version:

$$\rho c \frac{\partial T}{\partial t} = \frac{\partial e}{\partial t} = -\operatorname{div}(\vec{q}) = \operatorname{div}(\lambda \nabla T)$$

$$= \frac{\partial}{\partial x}\left(\lambda \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(\lambda \frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(\lambda \frac{\partial T}{\partial z}\right) \tag{9}$$

Simplified version for constant λ:

$$\rho c \frac{\partial T}{\partial t} = \lambda \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) = \lambda \Delta T \tag{10}$$

with the Laplace operator

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \operatorname{div}\nabla \tag{11}$$

Fundamentals – Heat Transport

The Thermal Diffusivity

The heat conduction equation for constant λ can be written in the form

$$\frac{\partial T}{\partial t} \;=\; \kappa \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right) \;=\; \kappa \, \Delta T \qquad (12)$$

with the thermal diffusivity

$$\kappa \;=\; \frac{\lambda}{\rho c}$$

Water: $\kappa = 1.4 \times 10^{-7} \, \frac{\text{m}^2}{\text{s}}$

Rocks: $\kappa \approx 10^{-6} \, \frac{\text{m}^2}{\text{s}} \approx 30 \, \frac{\text{m}^2}{\text{a}}$

Fundamentals – Heat Transport

Thermal Conductivity (λ)

Unit: $\frac{W}{m\,K}$

Meaning: Describes how well a material conducts heat.

Heat Capacity (c, c_{vol}, c_{mol})

Unit: $\frac{J}{kg\,K}$, $\frac{J}{m^3 K}$, $\frac{J}{mol\,K}$
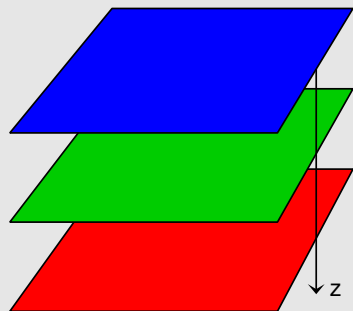
Meaning: Describes how much energy is needed to heat up a material.

Thermal Diffusivity (κ)

Unit: $\frac{m^2}{s}$

Meaning: Describes how rapidly temperature propagates.

The Temperature in Earth's Crust

One-Dimensional Description

Most of the large-scale heat conduction problems in the lithosphere can be approximated in 1D.

$T(x, y, z, t)$ does not depend on x and y

$$\rho c \frac{\partial}{\partial t} T(z, t) = -\frac{\partial}{\partial z} q(z, t) \qquad (13)$$

$$= \frac{\partial}{\partial z} \left(\lambda \frac{\partial}{\partial z} T(z, t) \right) \quad (14)$$

The z axis is often assumed to point in downward direction.

The Temperature in Earth's Crust

One-Dimensional Steady-State Geotherms

$$\rho c \frac{\partial}{\partial t} T(z,t) \; = \; - \frac{\partial}{\partial z} q(z,t) \; = \; 0 \tag{15}$$

$$\Downarrow$$

$$q(z) \; = \; - \lambda \frac{\partial}{\partial z} T(z) \; = \; - q_s \; = \; \text{const} \tag{16}$$

with $q_s = -q(0) =$ surface heat flux density

$$\Downarrow$$

$$T(z) \; = \; T_s + \frac{q_s}{\lambda} z \tag{17}$$

if λ is constant with $T_s = T(0) =$ surface temperature

The Temperature in Earth's Crust

Why are Geotherms Curved?

Spatial variation in λ: not very strong in general

Advective heat transport: important at some locations in the crust (hydrothermal circulation) and in the asthenosphere

Non-steady state: significant for oceanic crust and highly active continental crust

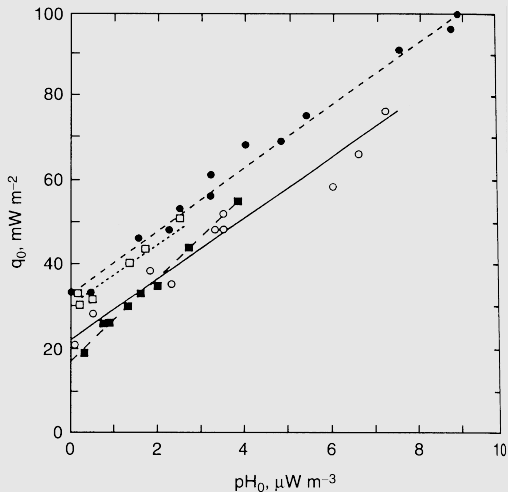Radiogenic heat production: strong effect in the upper continental crust

The Temperature in Earth's Crust

Radiogenic Heat Production

- Main contributions: decay of uranium ^{238}U, ^{235}U, thorium ^{232}Th, and potassium ^{40}K.
- Strong variation; typical heat production rates:

  granite: $S \approx 2.5 \, \frac{\mu W}{m^3}$

  basalt: $\quad S \approx 0.1 \, \frac{\mu W}{m^3}$

Correlation of Surface Heat Flux and Radiogenic Heat Production

Source: Turcotte & Schubert, Geodynamics

The Heat Conduction Equation with Heat Production

3D version:

$$\rho c \frac{\partial}{\partial t} T(\vec{x}, t) = -\text{div}\left(\vec{q}(\vec{x}, t)\right) + S(\vec{x}, t) \tag{18}$$

$$= \text{div}\left(\lambda \nabla T(\vec{x}, t)\right) + S(\vec{x}, t) \tag{19}$$

1D version:

$$\rho c \frac{\partial}{\partial t} T(z, t) = -\frac{\partial}{\partial z} q(z, t) + S(z, t) \tag{20}$$

$$= \frac{\partial}{\partial z}\left(\lambda \frac{\partial}{\partial z} T(z, t)\right) + S(z, t) \tag{21}$$

One-Dimensional Steady-State Geotherms

Steady-State Heat Conduction Equation with Heat Production

$$\frac{\partial}{\partial z} q(z) = S(z) \tag{22}$$

$$\Downarrow$$

$$q(z) - q(0) = \int_0^z S(\xi)\, d\xi \tag{23}$$

$$\Downarrow$$

$$q_s - q_b = -q(0) - (-q(d)) = \int_0^d S(\xi)\, d\xi \tag{24}$$

where
$$d = \text{thickness of the lithosphere}$$
$$q_s = -q(0) = \text{surface heat flux density}$$
$$q_b = -q(d) = \text{basal heat flux density}$$

One-Dimensional Steady-State Geotherms

The Relationship Between Heat Flux Density and Heat Production

From the diagram:

- Variations in surface heat flux density mainly arise from variations in radiogenic heat production.
- Straight line

$$q_s = q_b + hS_s \tag{25}$$

where S_0 = heat production rate at the surface.

Typical values:

$$q_b \approx 28 \, \tfrac{\text{mW}}{\text{m}^2} \tag{26}$$

$$h = \frac{q_s - q_b}{S_s} \approx 10 \, \text{km} \tag{27}$$

One-Dimensional Steady-State Geotherms

How is Radiogenic Heat Production Distributed in the Crust?

Two simple models:

Model 1: constant heat production down to a given depth h

$$S(z) = \begin{cases} S_s & \text{for } z \leq h \\ 0 & \text{else} \end{cases} \tag{28}$$

$$\Downarrow$$

$$q(z) = q(0) + \int_0^z S(\xi)\, d\xi = -\begin{cases} q_s - S_s z & \text{for } z \leq h \\ q_b & \text{else} \end{cases}$$

$$= -\begin{cases} q_s - (q_s - q_b)\frac{z}{h} & \text{for } z \leq h \\ q_b & \text{else} \end{cases} \tag{29}$$

One-Dimensional Steady-State Geotherms

How is Radiogenic Heat Production Distributed in the Crust?

Model 2: exponentially decreasing heat production rate

$$S(z) = S_s \, e^{-\frac{z}{h}} \tag{30}$$

$$\Downarrow$$

$$
\begin{aligned}
q(z) &= q(0) + \int_0^z S(\xi) \, d\xi = -\left(q_s - S_s h \left(1 - e^{-\frac{z}{h}}\right)\right) \\
&= -\left(q_b + (q_s - q_b) \, e^{-\frac{z}{h}}\right) \tag{31}
\end{aligned}
$$

- Both models cannot be distinguished from the surface data.
- Model 2 is theoretically better as it is consistent with surface erosion.

Time-Dependent Geotherms

Solution for a Sudden Change in Surface Temperature

Assume an initial temperature $T(z, 0) = 0$ ($z \geq 0$) and that the surface temperature switches to $T(0, t) = 1$ at $t = 0$.

No characteristic length scale and no characteristic time scale, but length scale and time scale are related by the thermal diffusivity κ.

$$\Downarrow$$

$$L(t) \;=\; \sqrt{\kappa t} \qquad (32)$$

defines a length scale for each time t.

Time-Dependent Geotherms

Solution for a Sudden Change in Surface Temperature

Idea: Assume that the shape of the temperature profile remains constant, while only the spatial scale changes.

$$\Downarrow$$

Look for solutions $T(z, t)$ which only depend on

$$u = \frac{z}{2L(t)} = \frac{z}{2\sqrt{\kappa t}} \tag{33}$$

instead of z and t indiviually. The factor 2 is only for convenience.

Time-Dependent Geotherms

Solution for a Sudden Change in Surface Temperature

$$\frac{\partial T}{\partial t} = \frac{\partial T}{\partial u}\frac{\partial u}{\partial t} = \frac{\partial T}{\partial u}\frac{z}{2\sqrt{\kappa}}\frac{-1}{2t^{\frac{3}{2}}} = \frac{\partial T}{\partial u}\frac{-u}{2t} \tag{34}$$

$$\frac{\partial T}{\partial z} = \frac{\partial T}{\partial u}\frac{\partial u}{\partial z} = \frac{\partial T}{\partial u}\frac{1}{2\sqrt{\kappa t}} \tag{35}$$

$$\frac{\partial^2 T}{\partial z^2} = \frac{\partial^2 T}{\partial u^2}\frac{1}{4\kappa t} \tag{36}$$

$$\Downarrow$$

$$\frac{\partial T}{\partial u}\frac{-u}{2t} = \kappa\frac{\partial^2 T}{\partial u^2}\frac{1}{4\kappa t} \tag{37}$$

$$\Downarrow$$

$$\frac{\partial}{\partial u}\frac{\partial T}{\partial u} = -2u\frac{\partial T}{\partial u} \tag{38}$$

Time-Dependent Geotherms

Solution for a Sudden Change in Surface Temperature

Solution:

$$\frac{\partial T}{\partial u} = a\,e^{-u^2} \tag{39}$$

with an arbitrary constant a.

$$\Downarrow$$

Solution for $T(u)$ with the condition $T(u) \to 0$ for $u \to \infty$:

$$T(u) = -\int\limits_{u}^{\infty} a\,e^{-x^2}\,dx \tag{40}$$

Time-Dependent Geotherms

Solution for a Sudden Change in Surface Temperature

Adjust a so that $T(0) = 1$:

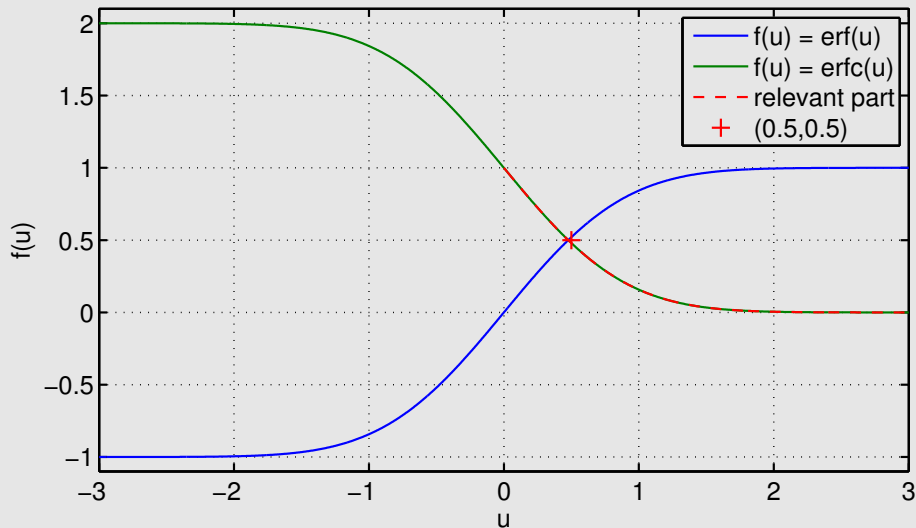$$T(u) = \frac{2}{\sqrt{\pi}} \int\limits_{u}^{\infty} e^{-x^2} dx = \text{erfc}(u) \tag{41}$$

where $\text{erfc}(u)$ is called complementary error function. It is related to the Gaussian error function

$$\text{erf}(u) = \frac{2}{\sqrt{\pi}} \int\limits_{0}^{u} e^{-x^2} dx \tag{42}$$

by

$$\text{erf}(u) + \text{erfc}(u) = 1 \tag{43}$$

Time-Dependent Geotherms

Solution for a Sudden Change in Surface Temperature

Time-Dependent Geotherms

Solution for a Sudden Change in Surface Temperature

$$T(z, t) \;=\; \operatorname{erfc}\left(\frac{z}{2L(t)}\right) \;=\; \operatorname{erfc}\left(\frac{z}{2\sqrt{\kappa t}}\right) \tag{44}$$

describes the temperature in the domain $z \geq 0$ with the initial temperature $T(z, 0) = 0$ where the surface temperature switches to $T(0, t) = 1$ at $t = 0$.

The length scale of heat conduction $L(t) = \sqrt{\kappa t}$ describes the depth where the temperature is $T(L(t), t) = \frac{1}{2}$ at the time t.

Fundamentals – Linear Differential Equations

Superposition of Solutions

The heat conduction equation is linear:

- If T is a solution, aT is also a solution for any (constant) factor a.
- If T_1 and T_2 are solutions, $T_1 + T_2$ is also a solution.

⬇

Solutions of the heat conduction equation can be superposed.

Fundamentals – Linear Differential Equations

Superposition of Solutions

Example: Assume that

- $T_m(\vec{x})$ is a solution of the steady-state equation with heat production

$$\mathrm{div}\left(\lambda \nabla T_m(\vec{x})\right) + S(\vec{x}) = 0 \tag{45}$$

- $T_t(\vec{x}, t)$ is a solution of the time-dependent equation without heat production

$$\rho c \frac{\partial}{\partial t} T_t(\vec{x}, t) = \mathrm{div}\left(\lambda \nabla T_t(\vec{x}, t)\right) \tag{46}$$

$\Downarrow$

$T(\vec{x}, t) = T_m(\vec{x}) + T_t(\vec{x}, t)$ is a solution of the full equation

$$\rho c \frac{\partial}{\partial t} T(\vec{x}, t) = \mathrm{div}\left(\lambda \nabla T(\vec{x}, t)\right) + S(\vec{x}) \tag{47}$$

Seasonal Variation of the Subsurface Temperature

Harmonic Solution

Assume a harmonic oscillation of the surface temperature ($z = 0$):

$$T(0, t) = \cos(\omega t) \qquad (48)$$

with
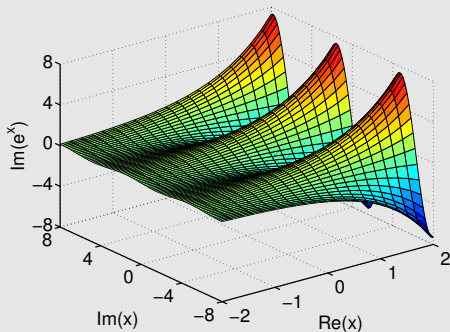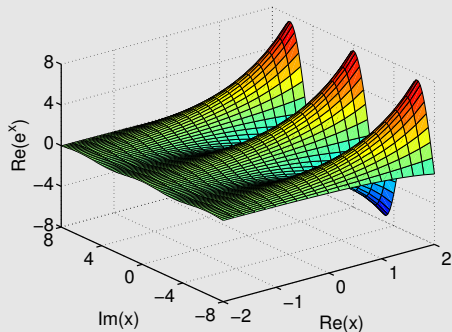
$$\omega = \frac{2\pi}{\tau} = \text{angular frequency}$$
$$\tau = \text{period (e.\,g., 1 year)}$$

Technically more convenient:

$$T(0, t) = e^{i\omega t} \qquad (49)$$

The Complex Exponential Function vs. Sine and Cosine

With $e^{i\phi} = \cos\phi + i\sin\phi$, the complex exponential function combines the real exponential function with the sine and cosine functions.

Seasonal Variation of the Subsurface Temperature

Harmonic Solution

Solve the differential equation

$$\frac{\partial}{\partial t} T(z, t) = \kappa \frac{\partial^2}{\partial z^2} T(z, t) \tag{50}$$

with $T(0, t) = e^{i\omega t}$. Assume that the harmonic oscillation persists for all depths z, but with a depth-dependent amplitude $A(z)$:

$$T(z, t) = e^{i\omega t} A(z) \tag{51}$$

Insert into Eq. 50:

$$\frac{\partial^2}{\partial z^2} A(z) = \frac{i\omega}{\kappa} A(z) \tag{52}$$

Solution:

$$A(z) = e^{\pm\sqrt{\frac{i\omega}{\kappa}} z} = e^{\pm(1+i)\sqrt{\frac{\omega}{2\kappa}} z} \tag{53}$$

Seasonal Variation of the Subsurface Temperature

Harmonic Solution

The condition $A(z) \to 0$ for $z \to \infty$ is only satisfied with the minus sign.

$$
\begin{aligned}
T(z, t) &= e^{i\omega t}\, e^{-(1+i)\sqrt{\frac{\omega}{2\kappa}}z} = e^{i\left(\omega t - \sqrt{\frac{\omega}{2\kappa}}z\right)}\, e^{-\sqrt{\frac{\omega}{2\kappa}}z} \\
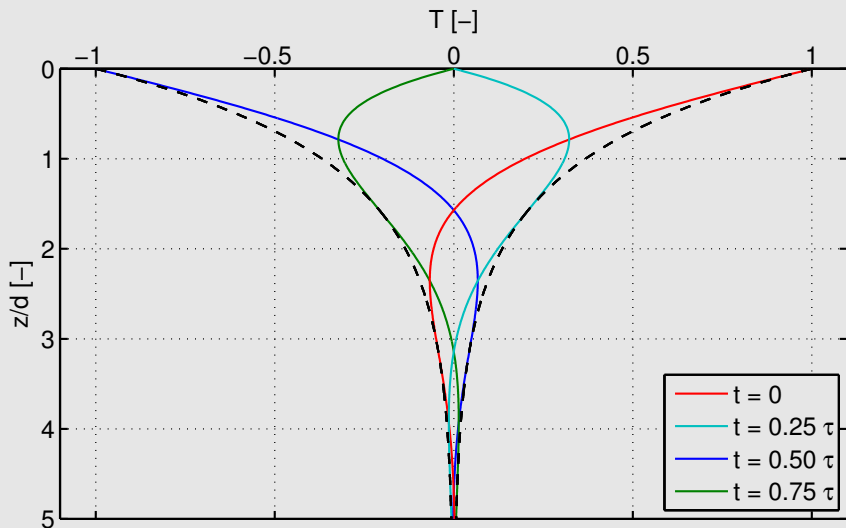&= e^{i\left(\omega t - \frac{z}{d}\right)} e^{-\frac{z}{d}}
\end{aligned}
\tag{54}
$$

with the depth of penetration

$$
d = \sqrt{\frac{2\kappa}{\omega}} = \sqrt{\frac{\kappa\tau}{\pi}}
\tag{55}
$$

With cosine instead of the complex exponential function:

$$
T(z, t) = \cos\left(\omega t - \frac{z}{d}\right) e^{-\frac{z}{d}}
\tag{56}
$$

Seasonal Variation of the Subsurface Temperature

Harmonic Solution

Harmonic Solution

Amplitude: $e^{-\frac{z}{d}}$ ➡ d is the depth where the temperature oscillation has decreased to $\frac{1}{e}$ of the oscillation at the surface.

Phase shift: Oscillation is opposite to the oscillation at the surface for $z = \pi d$.

The Diel Variation

Same equations as for the seasonal variation, but the depth of penetration d is almost 20 times lower.

Superposition of Seasonal and Diel Oscillation

$$T(z, t) \;=\; T_m(z) + T_y(z, t) + T_d(z, t) \tag{57}$$

with

$$
\begin{aligned}
T_m(z) &= \text{steady-state geotherm} \\[4pt]
T_y(z, t) &= a_y \cos\left(\omega_y(t - t_y) - \frac{z}{d_y}\right) e^{-\frac{z}{d_y}} \;=\; \text{seasonal variation} \tag{58} \\[4pt]
T_d(z, t) &= a_d \cos\left(\omega_d(t - t_d) - \frac{z}{d_d}\right) e^{-\frac{z}{d_d}} \;=\; \text{diel variation} \tag{59} \\[4pt]
a_y, a_d &= \text{amplitudes} \\
\omega_y, \omega_d &= \text{angular frequencies} \\
t_y, t_d &= \text{time lag of maximum temperature vs. } t = 0 \\
d_y, d_d &= \text{depths of penetration}
\end{aligned}
$$

Anharmonic Oscillations

Solution schemes:

- numerical simulation
- superposition of harmonic components with angular frequencies ω, 2ω, 3ω, ... (Fourier series)
- superposition of solutions for step-like variations

Solution approaches the harmonic solution for increasing depth.

Seasonal Variation of the Subsurface Temperature

Anharmonic Oscillations