

# Geothermics and Geothermal Energy

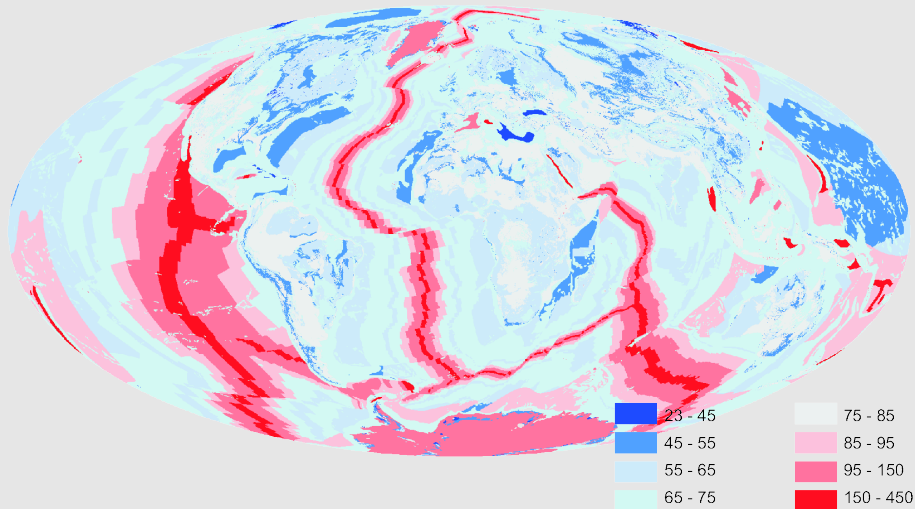
## The Temperature in Earth's Crust

Stefan Hergarten

Institut für Geo- und Umweltnaturwissenschaften  
Albert-Ludwigs-Universität Freiburg



## Regional Variation of the Surface Heat Flux Density



Source: Davies & Davies, Earth's surface heat flux, Solid Earth, 2010

## Definition of Heat Flux Density

$\vec{q}$  = energy per time and cross section area  $[\frac{W}{m^2}]$

## Earth's Mean Surface Heat Flux Density

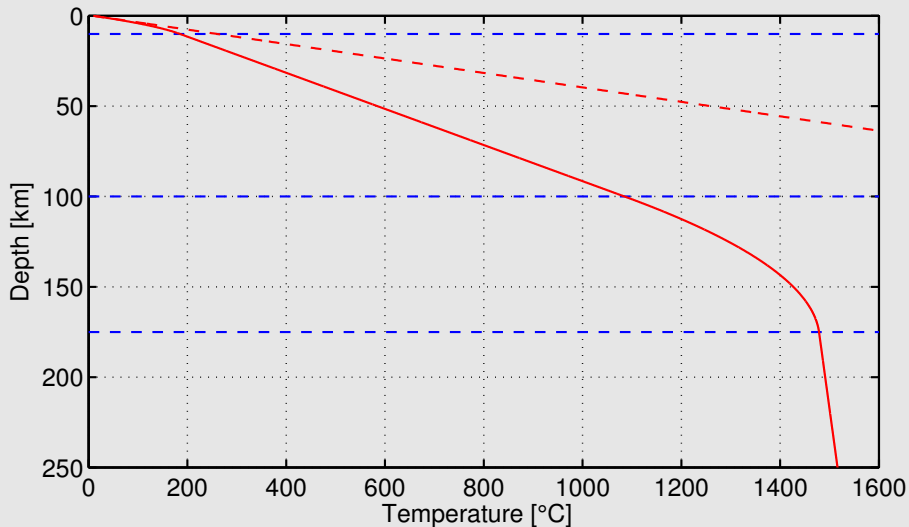
Continental crust (40 % of total surface):  $q = 70.9 \frac{mW}{m^2}$

Oceanic crust (60 % of total surface):  $q = 105.4 \frac{mW}{m^2}$

Overall mean:  $q = 91.6 \frac{mW}{m^2}$

- $q$  is more than 4 orders of magnitude smaller than the solar constant  $S = 1367 \frac{W}{m^2}$ .
- $q$  describes the long-term mean energy balance of the solid Earth.
- $q$  reflects the ongoing cooling of Earth and from radiogenic heat production in the upper continental crust.

## A Typical Continental Geotherm



## Fourier's Law of Heat Conduction (1822)

- 1 Heat flow follows the direction of steepest descent of the temperature field  $T(\vec{x}, t) = T(x, y, z, t)$ .
- 2 Heat flow is proportional to the decrease of temperature per length:

$$\vec{q}(\vec{x}, t) = -\lambda \nabla T(\vec{x}, t) = -\lambda \begin{pmatrix} \frac{\partial}{\partial x} T(x, y, z, t) \\ \frac{\partial}{\partial y} T(x, y, z, t) \\ \frac{\partial}{\partial z} T(x, y, z, t) \end{pmatrix} \quad (1)$$

with

$\vec{q}(\vec{x}, t)$  = heat flux density (energy per area and time)  $[\frac{W}{m^2}]$

$\lambda$  = thermal conductivity  $[\frac{W}{mK}]$

## The Thermal Conductivity

Typical Values:

Material	$\lambda \left[ \frac{\text{W}}{\text{mK}} \right]$
diamond	2300
iron	80
sand	0.6
polyethylene (PE)	0.48
expanded polystyrene (EPS)	0.033
water	0.6
air	0.026

Rocks	$\lambda \left[ \frac{\text{W}}{\text{mK}} \right]$
granite	2.8
basalt	2
dolomite	2.5
limestone	2.5
sandstone	2.5
clay	1.4
widely used value	2.5

## The Equation of Continuity (Energy Balance)

rate of change of the thermal energy contained in a given volume

=

energy per time entering at the boundaries

-

energy per time leaving at the boundaries

+

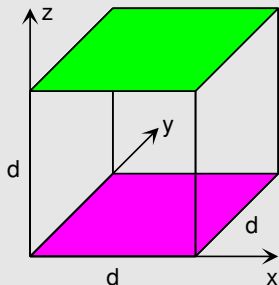
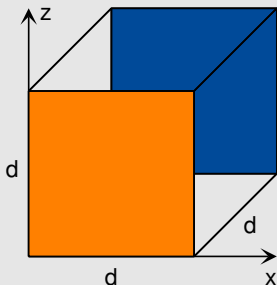
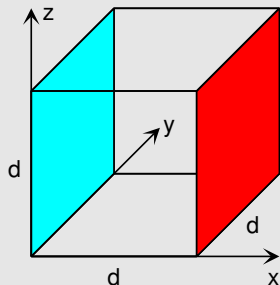
energy per time generated within the volume

-

energy per time consumed within the volume

## The Equation of Continuity (Energy Balance)

Energy balance of a cube without heat production:



Change in thermal energy  $E$  contained in the cube:

$$\frac{\partial E}{\partial t} = q_x d^2 - q_x d^2 + q_y d^2 - q_y d^2 + q_z d^2 - q_z d^2 \quad (2)$$



## The Equation of Continuity (Energy Balance)

Change in energy density  $e$  (thermal energy per volume):

$$\begin{aligned}\frac{\partial e}{\partial t} &= \frac{\partial E}{\partial t} \\ &= \frac{q_x - q_x}{d} + \frac{q_y - q_y}{d} + \frac{q_z - q_z}{d} \\ &\rightarrow -\frac{\partial q_x}{\partial x} - \frac{\partial q_y}{\partial y} - \frac{\partial q_z}{\partial z} \quad \text{for } d \rightarrow 0 \\ &= -\text{div}(\vec{q})\end{aligned}\tag{3}$$

with the divergence operator

$$\text{div}(\vec{q}) = \frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z}\tag{4}$$

## The Specific Heat Capacity

### Volumetric heat capacity

$$c_{\text{vol}} = \frac{\partial e}{\partial T} \quad (5)$$

describes the change in thermal energy density  $e$  [ $\frac{\text{J}}{\text{m}^3}$ ] with temperature  $T$ .

Unit:  $\frac{\text{J}}{\text{m}^3 \text{K}}$

### Specific heat capacity

$$c = \frac{c_{\text{vol}}}{\rho} = \frac{1}{\rho} \frac{\partial e}{\partial T} \quad (6)$$

is measured per mass instead of per volume ( $\rho = \text{density}$ ).

Unit:  $\frac{\text{J}}{\text{kg K}}$

## The Specific Heat Capacity

### Molar heat capacity

$$c_{\text{mol}} = Mc = \frac{M}{\rho} \frac{\partial e}{\partial T} \quad (7)$$

is measured per mol instead of per kg ( $M =$  molar mass).

Unit:  $\frac{\text{J}}{\text{mol K}}$

Dulong-Petit law:

$$c_{\text{mol}} \approx 3R \quad (8)$$

with the gas constant  $R = 8.314 \frac{\text{J}}{\text{mol K}}$  for most crystalline solids.

## The Specific Heat Capacity

Typical values at standard conditions:

Material	$c \left[ \frac{\text{J}}{\text{kg K}} \right]$
diamond	509
iron	450
sand	550
polyethylene (PE)	1250
expanded polystyrene (EPS)	1500
water	4187
air	1005

Rocks	$c \left[ \frac{\text{J}}{\text{kg K}} \right]$
granite	1000
basalt	850
dolomite	1000
limestone	900
sandstone	900
clay	1100
widely used value	900

## The Heat Conduction Equation (Energy Balance + Fourier's Law)

General version:

$$\begin{aligned}\rho c \frac{\partial T}{\partial t} &= \frac{\partial e}{\partial t} = -\operatorname{div}(\vec{q}) = \operatorname{div}(\lambda \nabla T) \\ &= \frac{\partial}{\partial x} \left( \lambda \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y} \left( \lambda \frac{\partial T}{\partial y} \right) + \frac{\partial}{\partial z} \left( \lambda \frac{\partial T}{\partial z} \right)\end{aligned}\quad (9)$$

Simplified version for constant  $\lambda$ :

$$\rho c \frac{\partial T}{\partial t} = \lambda \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right) = \lambda \Delta T \quad (10)$$

with the Laplace operator

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \operatorname{div} \nabla \quad (11)$$

## The Thermal Diffusivity

The heat conduction equation for constant  $\lambda$  can be written in the form

$$\frac{\partial T}{\partial t} = \kappa \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right) = \kappa \Delta T \quad (12)$$

with the thermal diffusivity

$$\kappa = \frac{\lambda}{\rho c}$$

Water:  $\kappa = 1.4 \times 10^{-7} \frac{\text{m}^2}{\text{s}}$

Rocks:  $\kappa \approx 10^{-6} \frac{\text{m}^2}{\text{s}} \approx 30 \frac{\text{m}^2}{\text{a}}$

## Thermal Conductivity ( $\lambda$ )

Unit:  $\frac{\text{W}}{\text{m K}}$

Meaning: Describes how well a material conducts heat.

## Heat Capacity ( $c$ , $c_{\text{vol}}$ , $c_{\text{mol}}$ )

Unit:  $\frac{\text{J}}{\text{kg K}}$ ,  $\frac{\text{J}}{\text{m}^3 \text{K}}$ ,  $\frac{\text{J}}{\text{mol K}}$

Meaning: Describes how much energy is needed to heat up a material.

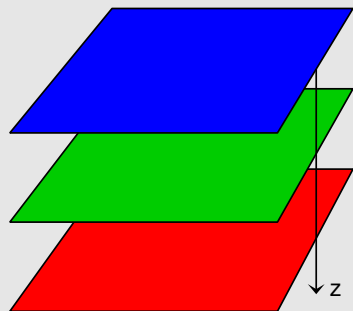
## Thermal Diffusivity ( $\kappa$ )

Unit:  $\frac{\text{m}^2}{\text{s}}$

Meaning: Describes how rapidly temperature propagates.

## One-Dimensional Description

Most of the large-scale heat conduction problems in the lithosphere can be approximated in 1D.



$T(x, y, z, t)$  does not depend on  $x$  and  $y$



$$\rho c \frac{\partial}{\partial t} T(z, t) = -\frac{\partial}{\partial z} q(z, t) \quad (13)$$

$$= \frac{\partial}{\partial z} \left( \lambda \frac{\partial}{\partial z} T(z, t) \right) \quad (14)$$

The  $z$  axis is often assumed to point in downward direction.



## One-Dimensional Steady-State Geotherms

$$\rho c \frac{\partial}{\partial t} T(z, t) = - \frac{\partial}{\partial z} q(z, t) = 0 \quad (15)$$



$$q(z) = - \lambda \frac{\partial}{\partial z} T(z) = - q_s = \text{const} \quad (16)$$

with  $q_s = -q(0) =$  surface heat flux density



$$T(z) = T_s + \frac{q_s}{\lambda} z \quad (17)$$

if  $\lambda$  is constant with  $T_s = T(0) =$  surface temperature

## Why are Geotherms Curved?

**Spatial variation in  $\lambda$ :** not very strong in general

**Advective heat transport:** important at some locations in the crust (hydrothermal circulation) and in the asthenosphere

**Non-steady state:** significant for oceanic crust and highly active continental crust

**Radiogenic heat production:** strong effect in the upper continental crust

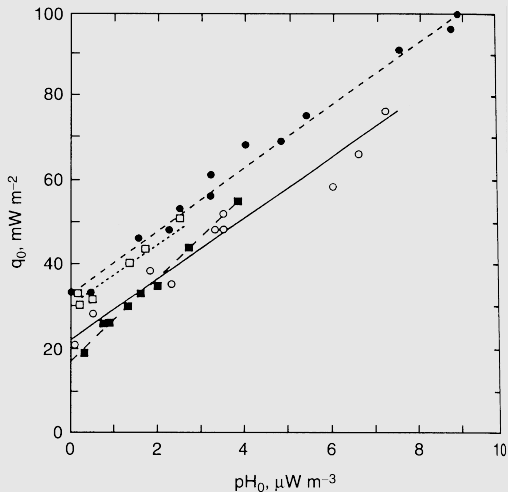
## Radiogenic Heat Production

- Main contributions: decay of uranium  $^{238}\text{U}$ ,  $^{235}\text{U}$ , thorium  $^{232}\text{Th}$ , and potassium  $^{40}\text{K}$ .
- Strong variation; typical heat production rates:

$$\text{granite: } S \approx 2.5 \frac{\mu\text{W}}{\text{m}^3}$$

$$\text{basalt: } S \approx 0.1 \frac{\mu\text{W}}{\text{m}^3}$$

## Correlation of Surface Heat Flux and Radiogenic Heat Production



Source: Turcotte & Schubert, Geodynamics

## The Heat Conduction Equation with Heat Production

3D version:

$$\rho c \frac{\partial}{\partial t} T(\vec{x}, t) = -\operatorname{div}(\vec{q}(\vec{x}, t)) + S(\vec{x}, t) \quad (18)$$

$$= \operatorname{div}(\lambda \nabla T(\vec{x}, t)) + S(\vec{x}, t) \quad (19)$$

1D version:

$$\rho c \frac{\partial}{\partial t} T(z, t) = -\frac{\partial}{\partial z} q(z, t) + S(z, t) \quad (20)$$

$$= \frac{\partial}{\partial z} \left( \lambda \frac{\partial}{\partial z} T(z, t) \right) + S(z, t) \quad (21)$$

## Steady-State Heat Conduction Equation with Heat Production

$$\frac{\partial}{\partial z} q(z) = S(z) \quad (22)$$



$$q(z) - q(0) = \int_0^z S(\xi) d\xi \quad (23)$$



$$q_s - q_b = -q(0) - (-q(d)) = \int_0^d S(\xi) d\xi \quad (24)$$

where

$d$  = thickness of the lithosphere

$q_s = -q(0)$  = surface heat flux density

$q_b = -q(d)$  = basal heat flux density

## The Relationship Between Heat Flux Density and Heat Production

From the diagram:

- Variations in surface heat flux density mainly arise from variations in radiogenic heat production.
- Straight line

$$q_s = q_b + hS_s \quad (25)$$

where  $S_0$  = heat production rate at the surface.

Typical values:

$$q_b \approx 28 \frac{\text{mW}}{\text{m}^2} \quad (26)$$

$$h = \frac{q_s - q_b}{S_s} \approx 10 \text{ km} \quad (27)$$

## How is Radiogenic Heat Production Distributed in the Crust?

Two simple models:

**Model 1:** constant heat production down to a given depth  $h$

$$S(z) = \begin{cases} S_s & \text{for } z \leq h \\ 0 & \text{else} \end{cases} \quad (28)$$



$$\begin{aligned} q(z) &= q(0) + \int_0^z S(\xi) d\xi = \begin{cases} q_s - S_s z & \text{for } z \leq h \\ q_b & \text{else} \end{cases} \\ &= \begin{cases} q_s - (q_s - q_b) \frac{z}{h} & \text{for } z \leq h \\ q_b & \text{else} \end{cases} \end{aligned} \quad (29)$$



## How is Radiogenic Heat Production Distributed in the Crust?

Model 2: exponentially decreasing heat production rate

$$S(z) = S_s e^{-\frac{z}{h}} \quad (30)$$



$$\begin{aligned} q(z) &= q(0) + \int_0^z S(\xi) d\xi = - \left( q_s - S_s h \left( 1 - e^{-\frac{z}{h}} \right) \right) \\ &= - \left( q_b + (q_s - q_b) e^{-\frac{z}{h}} \right) \end{aligned} \quad (31)$$

- Both models cannot be distinguished from the surface data.
- Model 2 is theoretically better as it is consistent with surface erosion.

## Analytical Solutions

A wealth of analytical solutions has been developed during the last centuries (see, e. g., Carslaw & Jaeger, Conduction of Heat in Solids).

- Restricted to specific situations
- In principle sufficient for all problems considered in this class.

## Numerical Approximations

Most widely used techniques:

- Finite differences
- Finite elements

## Separation of Variables

Look for solutions of the heat conduction equation

$$\frac{\partial}{\partial t} T(\vec{x}, t) = \kappa \Delta T(\vec{x}, t) \quad (32)$$

that can be written as a product

$$T(\vec{x}, t) = f(\vec{x}) g(t) \quad (33)$$



$$\frac{d}{dt} g(t) = \Lambda g(t) \quad \text{and} \quad \Delta f(\vec{x}) = \frac{\Lambda}{\kappa} f(\vec{x}) \quad (34)$$

with a constant  $\Lambda$ .

## Separation of Variables

Solution for  $g(t)$  with  $g(0) = 1$ :

$$g(t) = e^{\Lambda t} \quad (35)$$

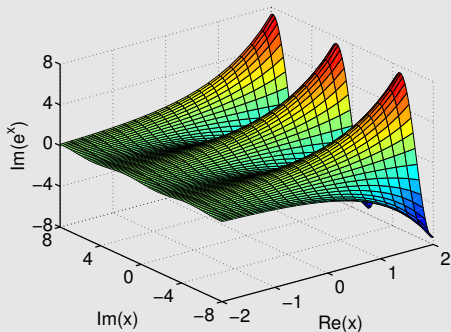
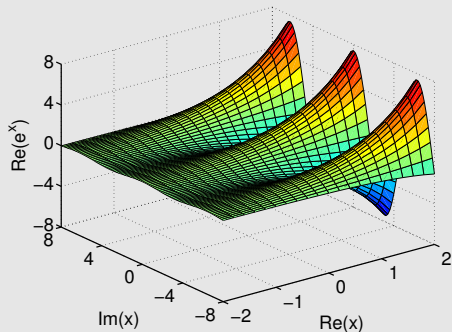
$\Lambda > 0$ : Temperature increases exponentially (practically impossible).

$\Lambda < 0$ : Temperature decreases exponentially.

**Most general solution:** complex number  $\Lambda$ ; combination of increase / decrease and harmonic oscillation

## The Complex Exponential Function

With  $e^{i\phi} = \cos \phi + i \sin \phi$ , the complex exponential function combines the real exponential function with the sine and cosine functions.



## Harmonic Solution

Assume a harmonic oscillation of the surface temperature ( $z = 0$ ):

$$T(0, t) = \cos(\omega t) \quad (36)$$

with

$$\omega = \frac{2\pi}{\tau} = \text{angular frequency}$$

$$\tau = \text{period (e. g., 1 year)}$$

Use the separation approach in 1D with  $\Lambda = i\omega$ :

$$T(z, t) = f(z) e^{i\omega t} \quad (37)$$



$$\frac{\partial^2}{\partial z^2} f(z) = \frac{\Lambda}{\kappa} f(z) = \frac{i\omega}{\kappa} f(z) \quad (38)$$

## Harmonic Solution

Solution with  $f(0) = 1$ :

$$f(z) = e^{\pm\sqrt{\frac{i\omega}{\kappa}}z} = e^{\pm(1+i)\sqrt{\frac{\omega}{2\kappa}}z} \quad (39)$$



$$T(z, t) = e^{i\omega t} e^{\pm(1+i)\sqrt{\frac{\omega}{2\kappa}}z} = e^{i(\omega t \pm \sqrt{\frac{\omega}{2\kappa}}z)} e^{\pm\sqrt{\frac{\omega}{2\kappa}}z} \quad (40)$$

Only the version with the minus sign is physically reasonable ( $T(z, t) \rightarrow 0$  for  $z \rightarrow \infty$ ).

## Harmonic Solution

$$d = \sqrt{\frac{2\kappa}{\omega}} = \sqrt{\frac{\kappa T}{\pi}} \quad (41)$$

defines the **depth of penetration**:

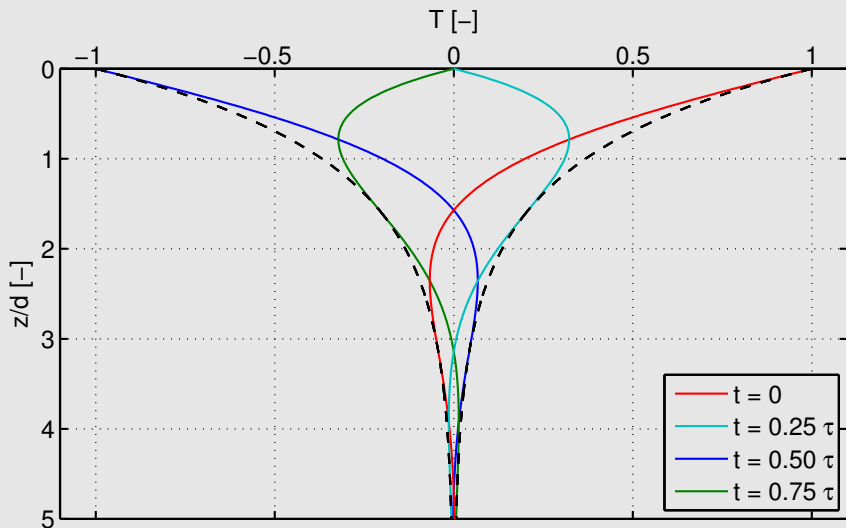
$$T(z, t) = e^{i(\omega t - \frac{z}{d})} e^{-\frac{z}{d}} \quad (42)$$

Solution with cosine instead of the complex exponential function:

$$T(z, t) = \cos\left(\omega t - \frac{z}{d}\right) e^{-\frac{z}{d}} \quad (43)$$



## Harmonic Solution



## Harmonic Solution

**Amplitude:**  $e^{-\frac{z}{d}}$  →  $d$  is the depth where the temperature oscillation has decreased to  $\frac{1}{e}$  of the oscillation at the surface.

**Phase shift:** Oscillation is opposite to the oscillation at the surface for  $z = \pi d$ .

## The Diel Variation

Same equations as for the seasonal variation, but the depth of penetration  $d$  is almost 20 times lower.

## Superposition of Solutions

The heat conduction equation

$$\rho c \frac{\partial}{\partial t} T(\vec{x}, t) = \operatorname{div}(\lambda \nabla T(\vec{x}, t)) + S(\vec{x}, t) \quad (44)$$

is linear.



Solutions can be superposed.

## Homogeneous and Inhomogeneous Linear Differential Equations

The heat conduction equation is

**homogeneous** for  $S(\vec{x}, t) = 0$  and

**inhomogeneous** for  $S(\vec{x}, t) \neq 0$ .

## Superposition of Solutions

- If  $T(\vec{x}, t)$  is a solution of the homogeneous equation, any multiple  $\alpha T(\vec{x}, t)$  is also a solution.
- If  $T_1(\vec{x}, t)$  and  $T_2(\vec{x}, t)$  are solutions of the homogeneous equation,  $T_1(\vec{x}, t) + T_2(\vec{x}, t)$  is also a solution.
- If  $T_h(\vec{x}, t)$  is a solution of the homogeneous equation and  $T_i(\vec{x}, t)$  a solution of the inhomogeneous equation,  $T_h(\vec{x}, t) + T_i(\vec{x}, t)$  is a solution of the inhomogeneous equation.

## Typical Application

Assume that

- $T_m(\vec{x})$  is a solution of the steady-state equation with heat production

$$\operatorname{div}(\lambda \nabla T_m(\vec{x})) + S(\vec{x}) = 0 \quad (45)$$

- $T_t(\vec{x}, t)$  is a solution of the time-dependent equation without heat production

$$\rho c \frac{\partial}{\partial t} T_t(\vec{x}, t) = \operatorname{div}(\lambda \nabla T_t(\vec{x}, t)) \quad (46)$$



$T(\vec{x}, t) = T_m(\vec{x}) + T_t(\vec{x}, t)$  is a solution of the full equation

$$\rho c \frac{\partial}{\partial t} T(\vec{x}, t) = \operatorname{div}(\lambda \nabla T(\vec{x}, t)) + S(\vec{x}) \quad (47)$$

## Superposition of Seasonal and Diel Oscillation

$$T(z, t) = T_m(z) + T_y(z, t) + T_d(z, t) \quad (48)$$

with

$T_m(z)$  = steady-state geotherm

$$T_y(z, t) = a_y \cos \left( \omega_y(t - t_y) - \frac{z}{d_y} \right) e^{-\frac{z}{d_y}} = \text{seasonal variation} \quad (49)$$

$$T_d(z, t) = a_d \cos \left( \omega_d(t - t_d) - \frac{z}{d_d} \right) e^{-\frac{z}{d_d}} = \text{diel variation} \quad (50)$$

$a_y, a_d$  = amplitudes

$\omega_y, \omega_d$  = angular frequencies

$t_y, t_d$  = time lag of maximum temperature vs.  $t = 0$

$d_y, d_d$  = depths of penetration

## Solution for a Sudden Change in Surface Temperature

Consider 1D heat conduction in the domain  $z \geq 0$  with

- initial temperature  $T(z, 0) = 0$  and
- surface temperature switches to  $T(0, t) = 1$  at  $t = 0$ .

Relevant for many technical applications and, e. g., for the cooling of oceanic lithosphere.

Variables  $z$  and  $t$  cannot be separated here, but a scaling relation between  $z$  and  $t$  can be used:

$$L(t) = \sqrt{\kappa t} \quad (51)$$

can be seen as a time-dependent length scale, called **length scale of heat conduction**.

## Solution for a Sudden Change in Surface Temperature

Assume that the temperature only depends on the nondimensional variable

$$u(z, t) = \frac{z}{2L(t)} = \frac{z}{2\sqrt{\kappa t}} \quad (52)$$

instead of  $z$  and  $t$  individually:

$$T(z, t) = F(u(z, t)) = F\left(\frac{z}{2\sqrt{\kappa t}}\right) \quad (53)$$

The factor 2 is only for convenience.

Interpretation: Shape of the temperature profile remains constant, while only the spatial scale changes.



## Solution for a Sudden Change in Surface Temperature

$$\frac{\partial}{\partial t} T(z, t) = \frac{\partial}{\partial u} F(u(z, t)) \frac{\partial}{\partial t} u(z, t) = \frac{\partial}{\partial u} F(u) \frac{-u}{2t} \quad (54)$$

$$\frac{\partial}{\partial z} T(z, t) = \frac{\partial}{\partial u} F(u(z, t)) \frac{\partial}{\partial z} u(z, t) = \frac{\partial}{\partial u} F(u) \frac{1}{2\sqrt{\kappa t}} \quad (55)$$

$$\frac{\partial^2}{\partial z^2} T(z, t) = \frac{\partial^2}{\partial u^2} F(u) \left( \frac{1}{2\sqrt{\kappa t}} \right)^2 = \frac{\partial^2}{\partial u^2} F(u) \frac{1}{4\kappa t} \quad (56)$$



$$\frac{\partial^2}{\partial u^2} F(u) = -2u \frac{\partial}{\partial u} F(u) \quad (57)$$

Define  $G(u) = \frac{\partial}{\partial u} F(u)$ :

$$\frac{\partial}{\partial u} G(u) = -2u G(u) \quad (58)$$

## Solution for a Sudden Change in Surface Temperature

Solution:

$$G(u) = a e^{-u^2} \quad (59)$$

with an arbitrary constant  $a$ .



Solution for  $F(u)$  with the condition  $F(u) \rightarrow 0$  for  $u \rightarrow \infty$  ( $T(z, 0) = 0$ ):

$$F(u) = - \int_u^{\infty} G(x) dx = - \int_u^{\infty} a e^{-x^2} dx \quad (60)$$

## Solution for a Sudden Change in Surface Temperature

Integral cannot be computed analytically, but can be expressed by the complementary error function

$$\operatorname{erfc}(u) = \frac{2}{\sqrt{\pi}} \int_u^{\infty} e^{-x^2} dx \quad (61)$$

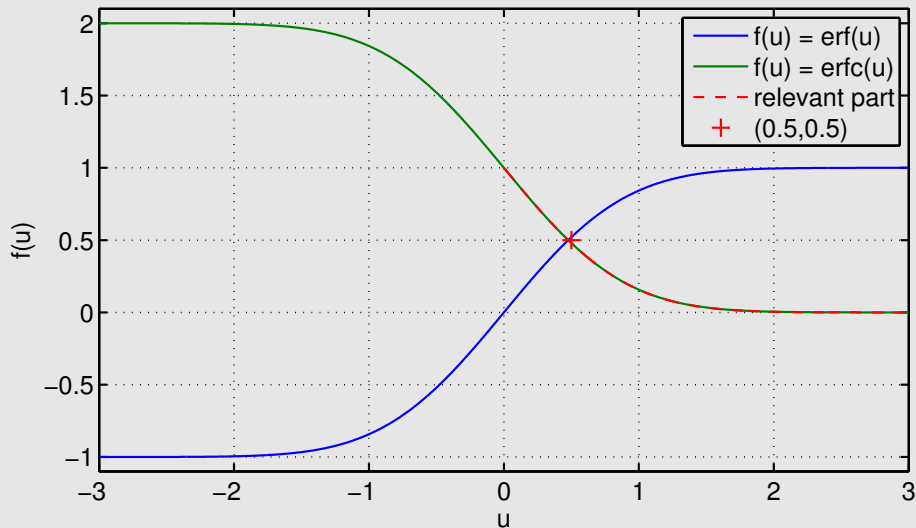
It is closely related to the Gaussian error function

$$\operatorname{erf}(u) = \frac{2}{\sqrt{\pi}} \int_0^u e^{-x^2} dx \quad (62)$$

by

$$\operatorname{erf}(u) + \operatorname{erfc}(u) = 1 \quad (63)$$

## Solution for a Sudden Change in Surface Temperature



## Solution for a Sudden Change in Surface Temperature

Solution satisfying the boundary condition  $F(0) = T(0, t) = 1$ :

$$F(u) = \operatorname{erfc}(u) \quad (64)$$



$$T(z, t) = \operatorname{erfc}\left(\frac{z}{2L(t)}\right) = \operatorname{erfc}\left(\frac{z}{2\sqrt{\kappa t}}\right) \quad (65)$$

describes the temperature in the domain  $z \geq 0$  with the initial temperature  $T(z, 0) = 0$  where the surface temperature switches to  $T(0, t) = 1$  at  $t = 0$ .

$L(t) = \sqrt{\kappa t}$  is the depth where the temperature is  $T(L(t), t) = \frac{1}{2}$  at the time  $t$ .