Seismology and Seismic Hazard

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Introduction

Seismology

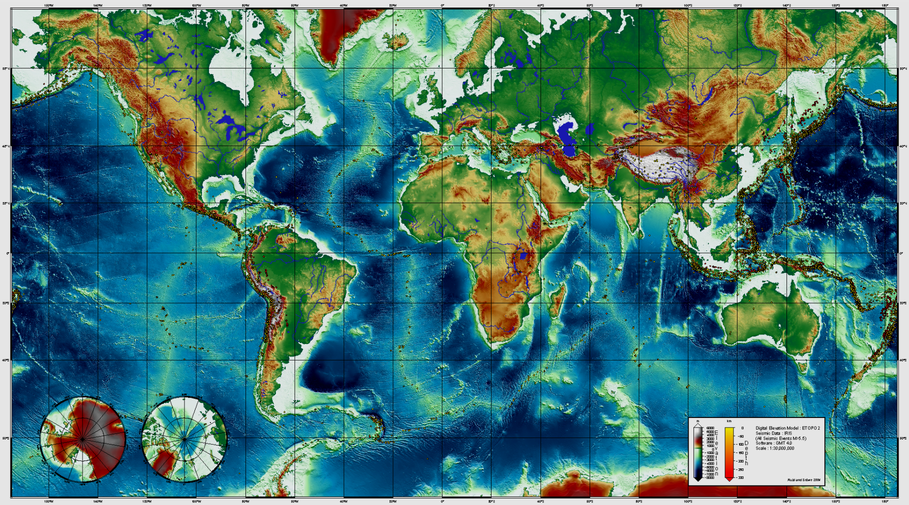
- Comprises all about earthquakes and the propagation of seismic waves in the Earth.
- One of the main fields of solid-earth geophysics.
- Has provided the majority of our knowledge on Earth's interior.

Seismics

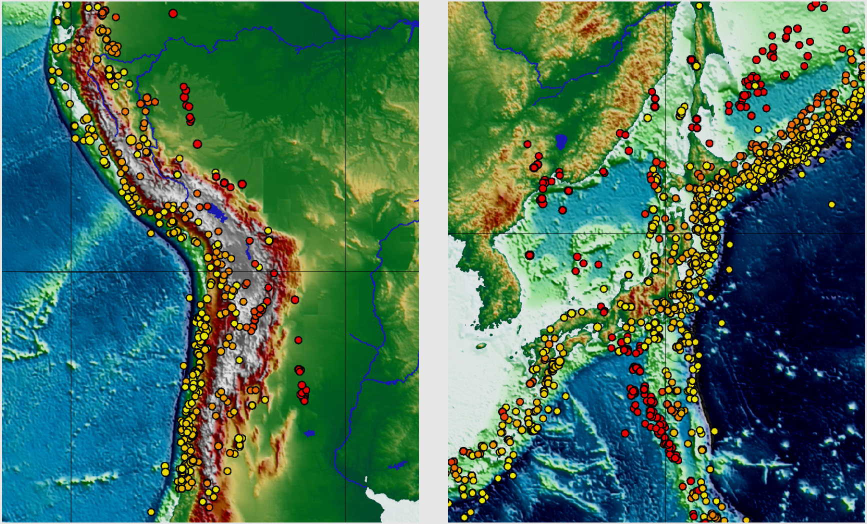
- Exploration of the deep and shallow subsurface with the help of artificial seismic waves.
- The perhaps most important field of applied geophysics.

Introduction

Worldwide Distribution of Earthquakes

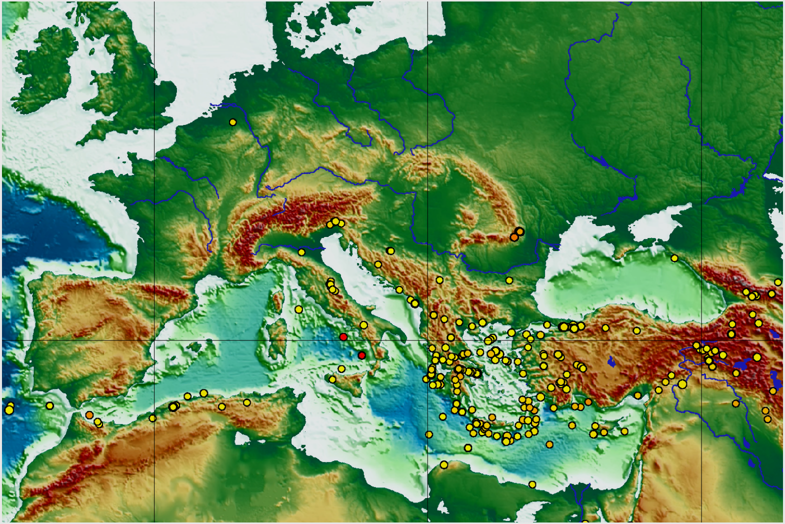


Worldwide Distribution of Earthquakes



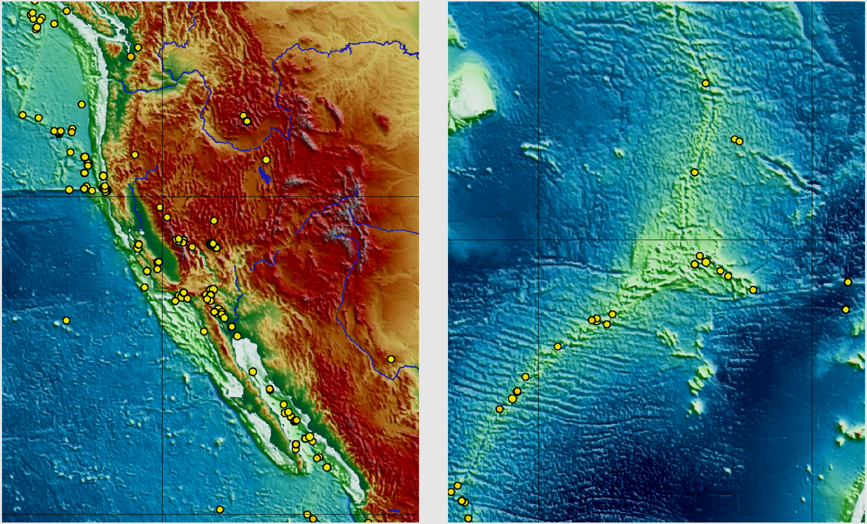
Introduction

Worldwide Distribution of Earthquakes



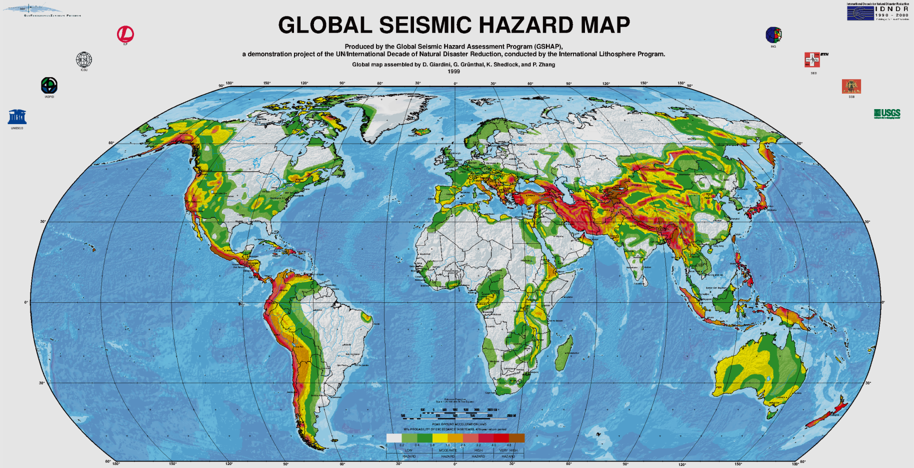
Introduction

Worldwide Distribution of Earthquakes



Introduction

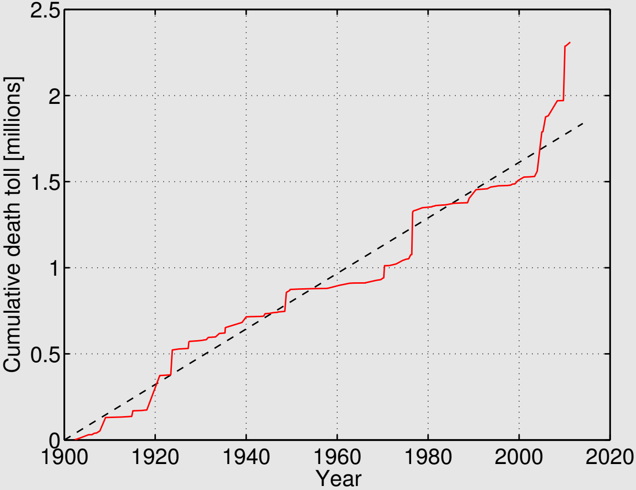
Earthquake Hazard



Source: Global Seismic Hazard Assessment Program

Introduction

Earthquakes with 1000 or more Deaths 1900–2014



Data: USGS Earthquake Hazards Program

The First "Seismometer" (132 a.D.)

Introduction

History of Seismology

1660	basic law of elasticity	R. Hooke
1821–22	differential equations of elasticity	C. Navier A. L. Cauchy
1830	theory of two fundamental types of elastic waves (P- and S-wave)	S. D. Poisson
1875	First "serious" seismometer	F. Gecchi
1887	theory of the first type of surface waves	J. W. Strutt (3. Lord Rayleigh)
1889	first recording of a distant earthquake	
1892	first compact seismometer, used at about 40 stations	J. Milne
1894	statistics of aftershocks	F. Omori
1903	12 degree scale for the intensity of earthquakes based on the damage	G. Mercalli

Introduction

History of Seismology

1906–1913	detection of the liquid core of the earth and determination of its size	R. D. Oldham, B. Gutenberg
1909	detection of the crust-mantle discontinuity	A. Mohorovičić
1911	theory of a second type of surface waves	A. E. H. Love
1935	local magnitude as an “objective” measure of earthquake intensity	C. F. Richter
1936	detection of the inner, solid core	I. Lehmann
1954	frequency-magnitude relation of earthquakes	B. Gutenberg, C. F. Richter
1975	first successful short-term prediction of a strong earthquake	
1977	moment magnitude as a measure of earthquake source strength	H. Kanamori

Theory of Seismic Waves

The Navier-Cauchy Equations in Seismology

- Small, but spatially and temporally variable displacement $\vec{u}(\vec{x}, t)$
- Neglect gravity
- Sign convention as in mathematics, physics, and engineering
- Elastic deformation

⬇

$$\rho \frac{\partial^2}{\partial t^2}\vec{u} \;=\; \begin{pmatrix} \frac{\partial\sigma_{11}}{\partial x_1} + \frac{\partial\sigma_{12}}{\partial x_2} + \frac{\partial\sigma_{13}}{\partial x_3} \\ \frac{\partial\sigma_{21}}{\partial x_1} + \frac{\partial\sigma_{22}}{\partial x_2} + \frac{\partial\sigma_{23}}{\partial x_3} \\ \frac{\partial\sigma_{31}}{\partial x_1} + \frac{\partial\sigma_{32}}{\partial x_2} + \frac{\partial\sigma_{33}}{\partial x_3} \end{pmatrix} \;=\; \mathrm{div}(\boldsymbol{\sigma}) \tag{1}$$

with the stress tensor

$$\boldsymbol{\sigma} \;=\; \lambda\,\epsilon_v\,\mathbf{1} + 2\mu\,\boldsymbol{\epsilon}, \tag{2}$$

The Navier-Cauchy Equations in Seismology

the strain tensor $\boldsymbol{\epsilon}$ consisting of the components

$$\epsilon_{ij} \;=\; \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right), \tag{3}$$

the volumetric strain

$$\epsilon_v \;=\; \epsilon_{11}+\epsilon_{22}+\epsilon_{33}, \tag{4}$$

the density ρ and the Lamé parameters of the medium λ and μ

One-Dimensional Wave Propagation

The Navier-Cauchy Equations in 1D

Displacement $u(x,t)$ instead of $\vec{u}(\vec{x},t)$.

↓

$$\rho \frac{\partial^2}{\partial t^2} u \;=\; \frac{\partial}{\partial x}\sigma \tag{5}$$

with

$$\sigma \;=\; (\lambda + 2\mu)\,\epsilon \;=\; (\lambda + 2\mu)\,\frac{\partial}{\partial x} u \tag{6}$$

↓

$$\rho \frac{\partial^2}{\partial t^2} u(x,t) \;=\; \frac{\partial}{\partial x}\left((\lambda + 2\mu)\,\frac{\partial}{\partial x} u(x,t) \right) \quad \text{(1D wave equation)} \tag{7}$$

One-Dimensional Wave Propagation

Solution of the 1D Wave Equation

If λ and μ are constant:

$$\rho \frac{\partial^2}{\partial t^2} u(x,t) = (\lambda + 2\mu) \frac{\partial^2}{\partial x^2} u(x,t) \tag{8}$$

Solution:

$$u(x,t) = f(t \pm sx) \tag{9}$$

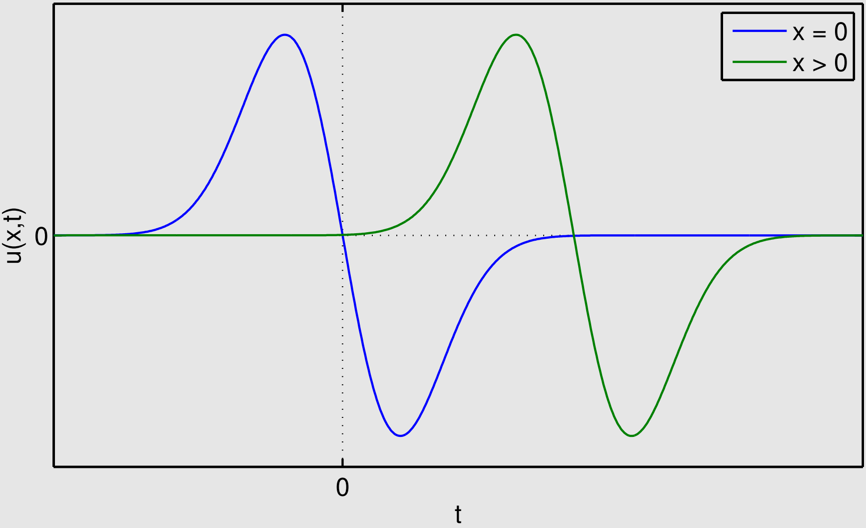
where

$$\begin{aligned} f &= \text{arbitrary function describing the shape of the wave} \\ s &= \sqrt{\frac{\rho}{\lambda + 2\mu}} = \text{slowness} \end{aligned}$$

The wave moves in positive or negative x direction with a velocity $v = \frac{1}{s}$.

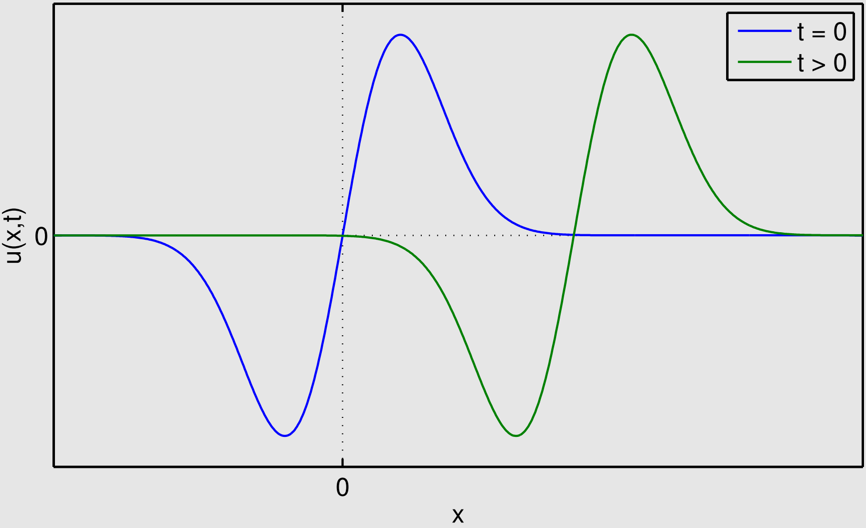
One-Dimensional Wave Propagation

Example



One-Dimensional Wave Propagation

Example



One-Dimensional Wave Propagation

The Retarded Time

$\tau = t \pm sx$ is called retarded time.

Meaning: At the position x and time t we observe what happened at the origin ($x = 0$, e. g. earthquake focus) at the retarded time $\tau = t \pm sx$.

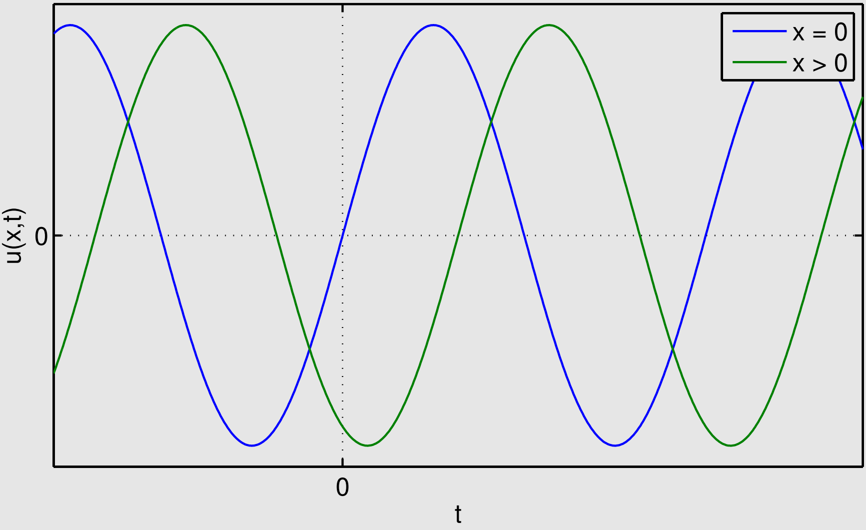
The Shape of the Wave

Examples for the function f:

- $f(\tau) = \begin{cases} 1 & \text{for } \tau \geq T \\ 0 & \text{else} \end{cases}$ describes a step-like shape (shock wave).
- $f(\tau) = \begin{cases} 1 & \text{for } |\tau| \leq \frac{T}{2} \\ 0 & \text{else} \end{cases}$ describes a boxcar-shaped wave.
- $f(\tau) = a\cos(\omega\tau)$, $f(\tau) = a\sin(\omega\tau)$ or $f(\tau) = a\,e^{i\omega\tau}$ describes a harmonic wave with an angular frequency ω and amplitude a.

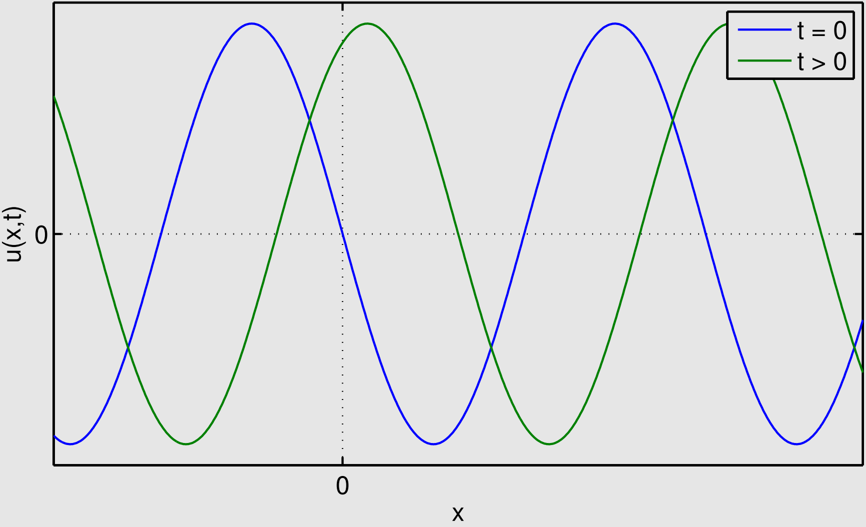
Harmonic Waves

Basic Terms



Harmonic Waves

Basic Terms



Harmonic Waves

Basic Terms

- Time domain:

  Angular frequency: $\omega\ [\frac{1}{\mathrm{s}}]$

  Frequency: $\nu = \frac{\omega}{2\pi}\ [\frac{1}{\mathrm{s}}]$

  Period: $T = \frac{1}{\nu} = \frac{2\pi}{\omega}\ [\mathrm{s}]$

- Spatial wave pattern:

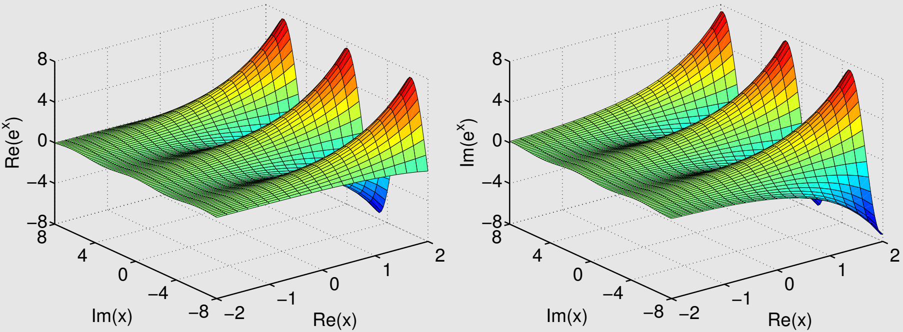
  Wave number: $k = \omega s\ [\frac{1}{\mathrm{m}}]$

  Wavelength: $L = \frac{2\pi}{k} = \frac{1}{\nu s}\ [\mathrm{m}]$

Harmonic Waves

The Complex Exponential Function vs. Sine and Cosine

With $e^{i\phi} = \cos\phi + i\sin\phi$, the complex exponential function combines the real exponential function with the sine and cosine functions.



Harmonic Waves

The Complex Exponential Function vs. Sine and Cosine

$$\mathrm{Re}\left(e^{i\omega\tau}\right) \;=\; \cos\left(\omega\tau\right) \quad \text{and} \quad \mathrm{Im}\left(e^{i\omega\tau}\right) \;=\; \sin\left(\omega\tau\right) \tag{10}$$

$$\Downarrow$$

Real part and imaginary part of the complex solution can be considered as independent real solutions.

Derivatives of the complex solutions are simpler than those of the real solutions:

$$\frac{\partial}{\partial\tau} e^{i\omega\tau} \;=\; i\omega e^{i\omega\tau} \tag{11}$$

while

$$\frac{\partial}{\partial\tau}\cos\left(\omega\tau\right) \;=\; -\,\omega\sin\left(\omega\tau\right) \quad \text{and} \quad \frac{\partial}{\partial\tau}\sin\left(\omega\tau\right) \;=\; \omega\cos\left(\omega\tau\right) \tag{12}$$

Seismic Body Waves

Seismic Waves in 3D

Complications towards the 1D case:

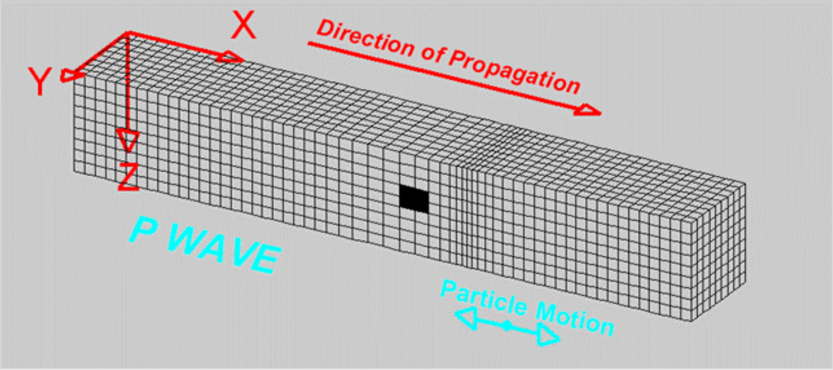
- Displacement $\vec{u}(\vec{x}, t)$ is a vector.
- Propagation in arbitrary direction in space instead of the positive or negative x axis only.

Seismic Body Waves

Fundamental Types of Body Waves

Two types of independent plane waves in an infinite, homogeneous elastic medium:

- Compressional wave (longitudinal wave, primary wave)

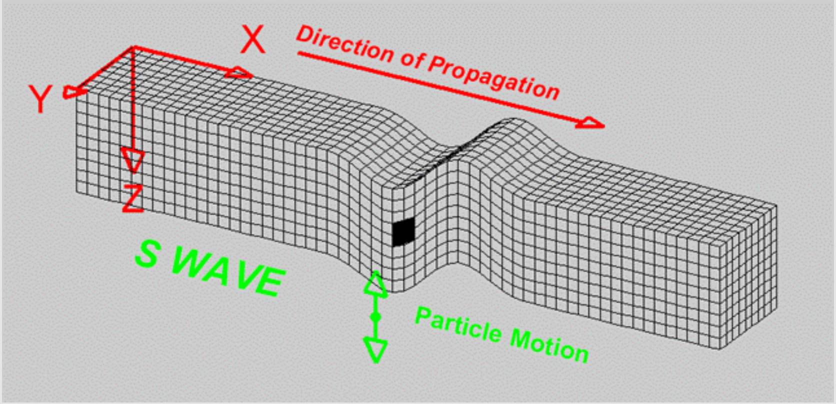


Source: L. Braile, Purdue University

Seismic Body Waves

Fundamental Types of Body Waves

- Shear wave (transverse wave, secondary wave)



Source: L. Braile, Purdue University

Seismic Body Waves

Plane Waves in Infinite, Homogeneous, and Isotropic Media

Plane wave: $\vec{u}(\vec{x}, t)$ is constant on parallel planes.

Mathematical description:

$$\vec{u}(\vec{x}, t) \;=\; f(t - \vec{s} \cdot \vec{x})\, \vec{a} \tag{13}$$

or for a harmonic wave:

$$\vec{u}(\vec{x}, t) \;=\; e^{i\omega(t - \vec{s} \cdot \vec{x})}\, \vec{a} \tag{14}$$

where

$\vec{s}$ = slowness vector

$\vec{a}$ = amplitude vector (constant)

The wave moves in direction of $\vec{s}$ with a velocity $v = \frac{1}{|\vec{s}|}$.

Plane Waves in Infinite, Homogeneous, and Isotropic Media

Simplest version: propagation in x_1 direction, $\vec{s} = \begin{pmatrix} s \\ 0 \\ 0 \end{pmatrix}$

$$\Downarrow$$

$$\vec{u}(\vec{x}, t) \;=\; f(t - sx_1)\,\vec{a} \tag{15}$$

$$\Downarrow$$

$$\epsilon \;=\; -\,s\,f'(t - sx_1) \begin{pmatrix} a_1 & \frac{1}{2}a_2 & \frac{1}{2}a_3 \\ \frac{1}{2}a_2 & 0 & 0 \\ \frac{1}{2}a_3 & 0 & 0 \end{pmatrix} \tag{16}$$

$$\Downarrow$$

$$\boldsymbol{\sigma} \;=\; -\,s\,f'(t - sx_1) \begin{pmatrix} (\lambda + 2\mu)a_1 & \mu a_2 & \mu a_3 \\ \mu a_2 & \lambda a_1 & 0 \\ \mu a_3 & 0 & \lambda a_1 \end{pmatrix} \tag{17}$$

Seismic Body Waves

Plane Waves in Infinite, Homogeneous, and Isotropic Media

If λ and μ are constant:

$$\operatorname{div}(\boldsymbol{\sigma}) \;=\; s^2 f''(t - sx_1)\begin{pmatrix}(\lambda+2\mu)a_1\\ \mu a_2\\ \mu a_3\end{pmatrix} \tag{18}$$

Insert into the Navier-Cauchy equations:

$$\rho\frac{\partial^2}{\partial t^2}\vec{u} \;=\; \rho f''(t - sx_1)\,\vec{a} \tag{19}$$

$$\;=\; \operatorname{div}(\boldsymbol{\sigma}) \;=\; s^2 f''(t - sx_1)\begin{pmatrix}(\lambda+2\mu)a_1\\ \mu a_2\\ \mu a_3\end{pmatrix} \tag{20}$$

Can only be satisfied if $a_2 = a_3 = 0$ and $s = \sqrt{\frac{\rho}{\lambda+2\mu}}$ (longitudinal polarization) or $a_1 = 0$ and $s = \sqrt{\frac{\rho}{\mu}}$ (transverse polarization).

Seismic Body Waves

Plane Waves in Infinite, Homogeneous, and Isotropic Media

General case (assignment 2): Navier-Cauchy equations can be satisfied only if either $\vec{a}$ is parallel (or opposite) to $\vec{s}$ or normal to $\vec{s}$.

Transverse wave: $\vec{a}$ is normal to $\vec{s}$ $(\vec{a} \cdot \vec{s} = 0)$

$$|\vec{s}|^2 = \frac{\rho}{\mu}, \quad v_s = \frac{1}{|\vec{s}|} = \sqrt{\frac{\mu}{\rho}} \tag{21}$$

Longitudinal wave: $\vec{a}$ is parallel or opposite to $\vec{s}$

$$|\vec{s}|^2 = \frac{\rho}{\lambda + 2\mu}, \quad v_p = \frac{1}{|\vec{s}|} = \sqrt{\frac{\lambda + 2\mu}{\rho}} \tag{22}$$

Seismic Body Waves

Comparison with Sound Waves in Liquids and Gases

The longitudinal wave is similar to sound waves in liquids and gases, while the transverse wave has no counterpart in liquids and gases.

Seismic Velocities

Medium	Longitudinal wave $[\frac{km}{s}]$	Transverse wave $[\frac{km}{s}]$
air	0.34	–
water	1.45	–
wood	about 3	about 1.8
Earth*	5.8–13.7	3.4–7.2

*Parametric Earth Models (PEM), not valid for the shallow subsurface

Seismic Body Waves

Seismic Velocities

$$\frac{v_p}{v_s} = \sqrt{\frac{\lambda + 2\mu}{\mu}} \geq \sqrt{\frac{4}{3}} \approx 1.15 \tag{23}$$

⬇

Velocity v_p of the longitudinal wave is always higher than the velocity v_s of the transverse wave.

⬇

Longitudinal wave always arrives prior to the transverse wave.

⬇

longitudinal wave = primary wave (P-wave)
transverse wave = secondary wave (S-wave)

Seismic Body Waves

Typical v_p-v_s Ratios

- For solid rocks:

$$\frac{v_p}{v_s} = \sqrt{\frac{\lambda + 2\mu}{\mu}} \approx \sqrt{3} \approx 1.7 \tag{24}$$

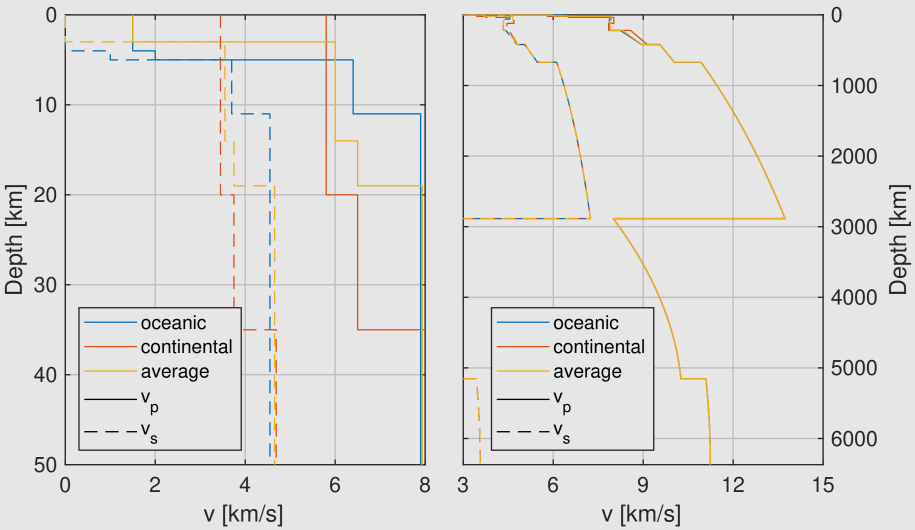
  for $\lambda \approx \mu$.

- For soil or unconsolidated rocks:

$$\frac{v_p}{v_s} \approx 2.5 \tag{25}$$

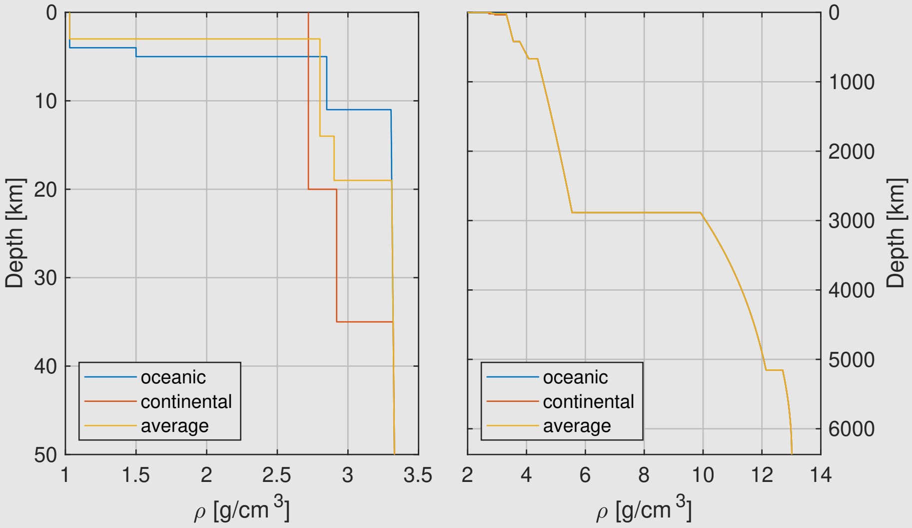
Seismic Body Waves

Seismic Velocities according to the Parametric Earth Models (PEM)



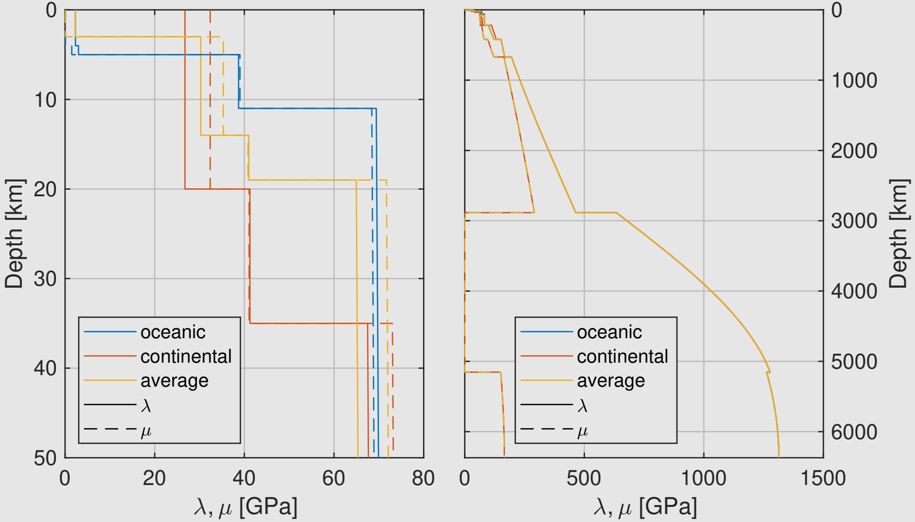
Seismic Body Waves

Density according to the Parametric Earth Models (PEM)



Seismic Body Waves

Lamé Parameters according to the Parametric Earth Models (PEM)



Seismic Body Waves

Typical P-wave Velocities in the Shallow Subsurface

Medium	v_p $[\frac{\text{km}}{\text{s}}]$
weathering zone	0.1–0.5
dry sand	0.3–0.6
water-saturated sand	1.3–1.8
sandstone	1.8–4
pit coal	1.6–1.9

Medium	v_p $[\frac{\text{km}}{\text{s}}]$
clay	1.2–2.8
claystone	2.2–4.2
limestone	3–6
halite	4.5–6.5
granite	5–6.5

The Point-Force Solution

- Infinite, homogeneous medium with parameters ρ, λ, and μ (like plane wave consideration).
- Assume that a given force $\vec{F}(t)$ acts at the origin ($\vec{x} = \vec{0}$).

Respective solution of the Navier-Cauchy equations:

$$\begin{aligned}\vec{u}_\mathrm{f}(\vec{x}, t) &= \frac{s_p^2}{4\pi\rho r}\mathbf{P}\vec{F}(t - s_p r) + \frac{s_s^2}{4\pi\rho r}\left(\mathbf{1} - \mathbf{P}\right)\vec{F}(t - s_s r) && (26)\\ &\quad + \frac{1}{4\pi\rho r^3}\left(3\mathbf{P} - \mathbf{1}\right)\int_{s_p r}^{s_s r} \tau\vec{F}(t - \tau)d\tau && (27)\end{aligned}$$

where

$$\begin{aligned} r &= |\vec{x}| \\ \mathbf{P} &= \vec{e}\,\vec{e}^{\,T} = \text{projection on radial direction}, \quad \vec{e} = \frac{\vec{x}}{r}\end{aligned}$$

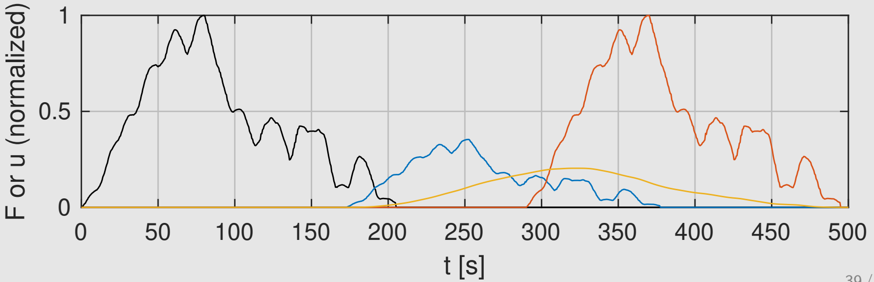
Earthquake Source Theory

The Point-Force Solution

$$\vec{u}_{\mathrm{f}}(\vec{x}, t) = \frac{s_p^2}{4\pi\rho r}\mathbf{P}\vec{F}(t - s_p r) + \frac{s_s^2}{4\pi\rho r}(\mathbf{1} - \mathbf{P})\vec{F}(t - s_s r) \qquad (28)$$

$$+\frac{1}{4\pi\rho r^3}(3\mathbf{P} - \mathbf{1})\int_{s_p r}^{s_s r}\tau\vec{F}(t-\tau)d\tau \qquad (29)$$

Example:



Earthquake Source Theory

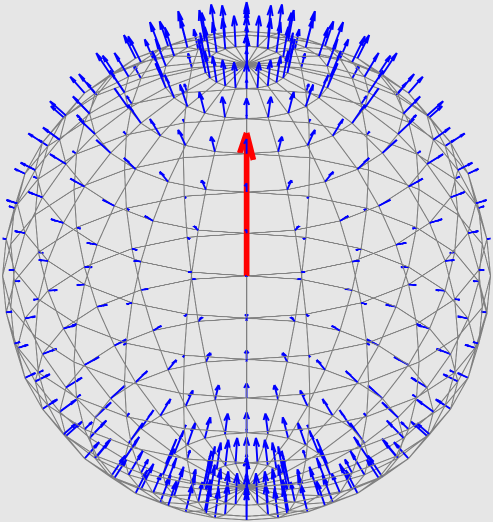
The Point-Force Solution

Spatial pattern of the first term,

$$\mathbf{P}\vec{F},$$

for

$$\vec{F} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$



Earthquake Source Theory

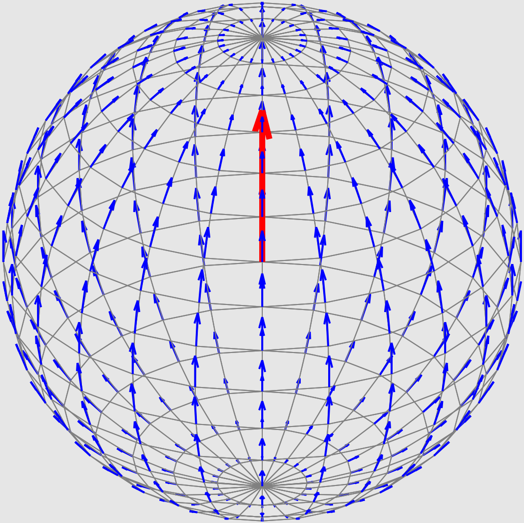
The Point-Force Solution

Spatial pattern of the second term,

$$(\mathbf{1} - \mathbf{P})\,\vec{F},$$

for

$$\vec{F} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$



Earthquake Source Theory

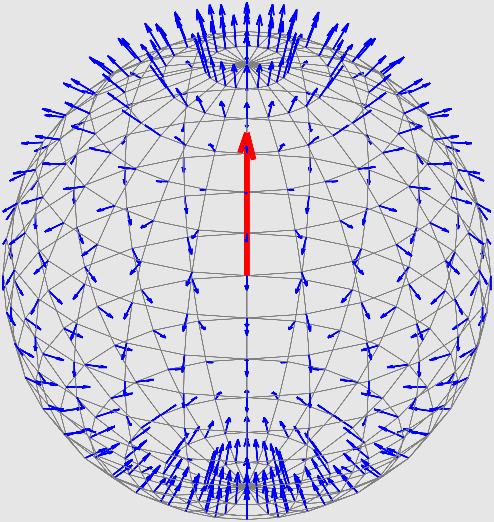
The Point-Force Solution

Spatial pattern of the third term,

$$(3\mathbf{P} - \mathbf{1})\,\vec{F}, \qquad (30)$$

for

$$\vec{F} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

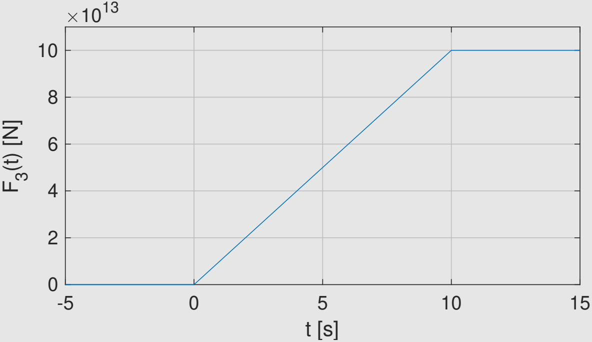


Earthquake Source Theory

The Point-Force Solution

Example: second term (S-wave) for

$$\vec{F}(t) = \begin{pmatrix} 0 \\ 0 \\ F_3(t) \end{pmatrix}$$



Earthquake Source Theory

Force Couples

Solution for a single point force causes an overall displacement in direction of the force.

↓

not possible

↓

Consider a couple of opposite forces $\vec{F}$ and $-\vec{F}$ displaced by a small vector $\vec{a}$ (at $\frac{\vec{a}}{2}$ and $-\frac{\vec{a}}{2}$).

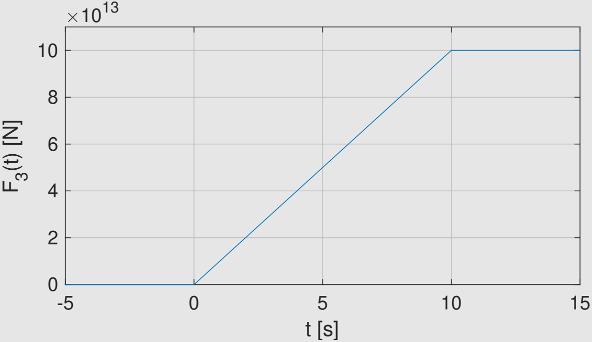
↓

$$\vec{u}(\vec{x}, t) \;=\; \vec{u}_{\mathrm{f}}(\vec{x} - \tfrac{\vec{a}}{2}, t) - \vec{u}_{\mathrm{f}}(\vec{x} + \tfrac{\vec{a}}{2}, t) \qquad (31)$$

The Point-Force Solution

Example: second term (S-wave) for

$$\vec{F}(t) = \begin{pmatrix} 0 \\ 0 \\ F_3(t) \end{pmatrix}, \quad \vec{a} = \begin{pmatrix} 100\,\mathrm{m} \\ 0 \\ 0 \end{pmatrix}$$



The Seismic Moment Tensor

Approximation in the limit $|\vec{a}| \to 0$:

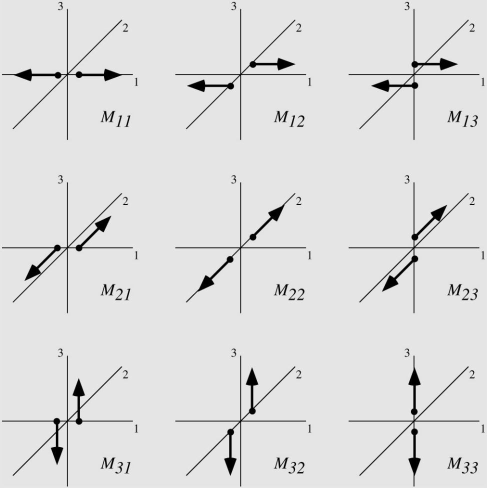
$$\begin{aligned}\vec{u}(\vec{x},t) &\approx -\nabla \vec{u}_{\mathrm{f}}(\vec{x},t)\vec{a} && (32)\\ &= -\mathrm{div}\left(\frac{s_p^2}{4\pi\rho r}\mathbf{P}\mathbf{M}(t-s_p r) + \frac{s_s^2}{4\pi\rho r}\left(\mathbf{1}-\mathbf{P}\right)\mathbf{M}(t-s_s r)\right.\\ &\qquad \left. + \frac{1}{4\pi\rho r^3}\left(3\mathbf{P}-\mathbf{1}\right)\int_{s_p r}^{s_s r}\tau\mathbf{M}(t-\tau)d\tau\right) && (33)\end{aligned}$$

with the seismic moment tensor (centroid moment tensor, CMT)

$$\mathbf{M}(t) = \vec{F}(t)\,\vec{a}^T \quad [Nm] \qquad (34)$$

Earthquake Source Theory

Components of the Seismic Moment Tensor



Source: Shearer, Introduction to Seismology

Earthquake Source Theory

The Permanent Displacement

Perform the derivatives (div) in Eq. 33.

↓

Terms proportional to $\mathbf{M}$ and terms proportional to $\dot{\mathbf{M}} = \frac{d}{dt}\mathbf{M}$

Terms proportional to $\mathbf{M}$:

- Persist after the earthquake has terminated, i. e., after $\mathbf{M}$ has become constant.

  ↓

  Describe the permanent displacement caused by the earthquake.
- Decrease like $\frac{1}{r^2}$ or faster.

Earthquake Source Theory

The Transient Displacement

Terms proportional to $\dot{\mathbf{M}}$:

- Arise from applying the chain rule to terms of the form $\frac{\partial}{\partial x_i}\mathbf{M}(t - sr)$.
- Vanish after the earthquake has terminated.

↓

Seismic waves

$$\vec{u}(\vec{x}, t) \; = \; \frac{s_p^3}{4\pi\rho r}\mathbf{P}\,\underset{\substack{\uparrow\\ \text{P-wave}}}{\dot{\mathbf{M}}}(t - s_p r)\,\vec{e} + \frac{s_s^3}{4\pi\rho r}\,(\mathbf{1} - \mathbf{P})\,\underset{\substack{\uparrow\\ \text{S-wave}}}{\dot{\mathbf{M}}}(t - s_s r)\,\vec{e} \qquad (35)$$

with $\vec{e} \; = \; \frac{\vec{x}}{r}$ and $\mathbf{P} = \vec{e}\,\vec{e}^{\,T}$

Earthquake Source Theory

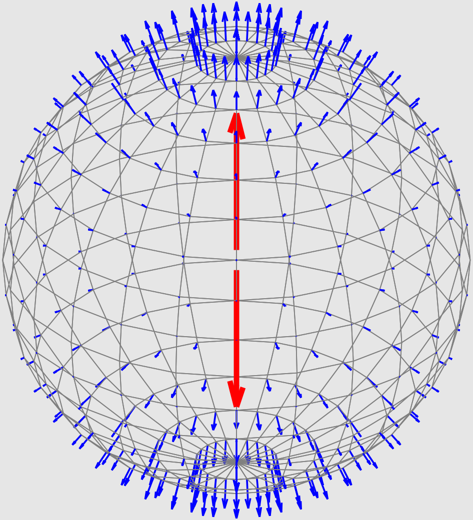
Seismic Wave Patterns

P-wave radiation pattern

$$\mathbf{P}\dot{\mathbf{M}}\vec{e}$$

for

$$\dot{\mathbf{M}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



Earthquake Source Theory

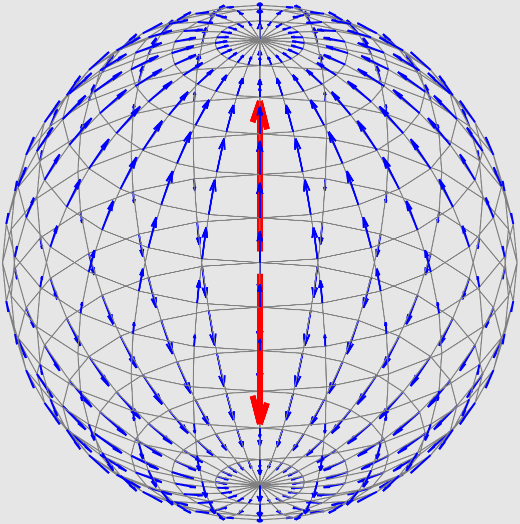
Seismic Wave Patterns

S-wave radiation pattern

$$(\mathbf{1}-\mathbf{P})\,\dot{\mathbf{M}}\vec{e}$$

for

$$\dot{\mathbf{M}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



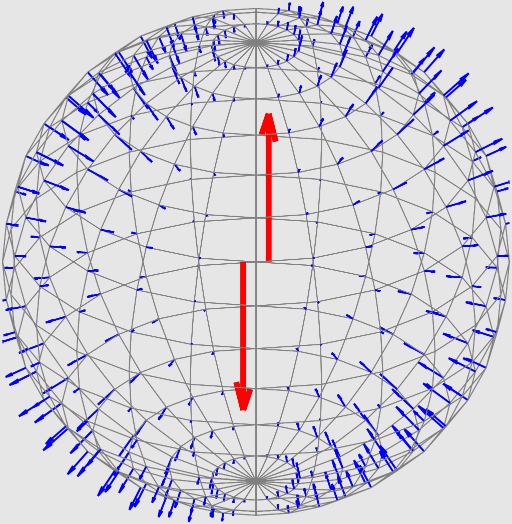
Seismic Wave Patterns

P-wave radiation pattern

$$\mathbf{P}\dot{\mathbf{M}}\vec{e}$$

for

$$\dot{\mathbf{M}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$



Earthquake Source Theory

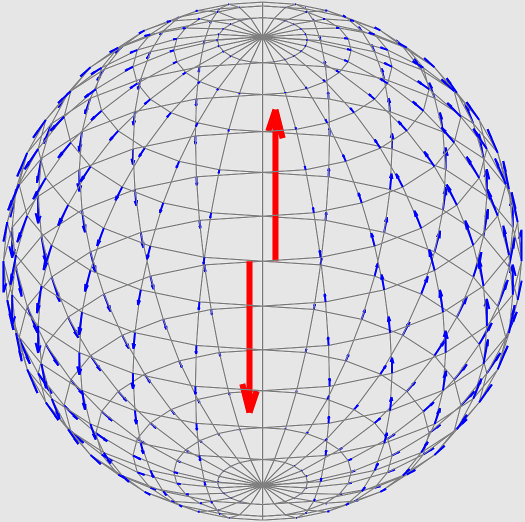
Seismic Wave Patterns

S-wave radiation pattern

$$(\mathbf{1}-\mathbf{P})\,\dot{\mathbf{M}}\vec{e}$$

for

$$\dot{\mathbf{M}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$



Symmetry of the Seismic Moment Tensor

Nondiagonal components of **M** cause an overall rotation.

↓

not possible

↓

M must be symmetric: $M_{ji} = M_{ij}$.

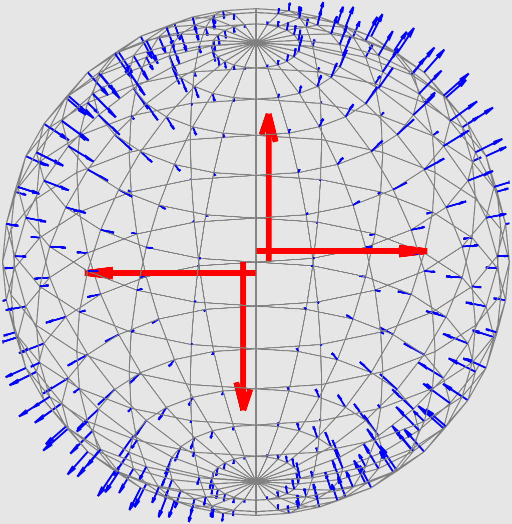
Seismic Wave Patterns

P-wave radiation pattern

$$\mathbf{P}\dot{\mathbf{M}}\vec{e}$$

for

$$\dot{\mathbf{M}} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$



Earthquake Source Theory

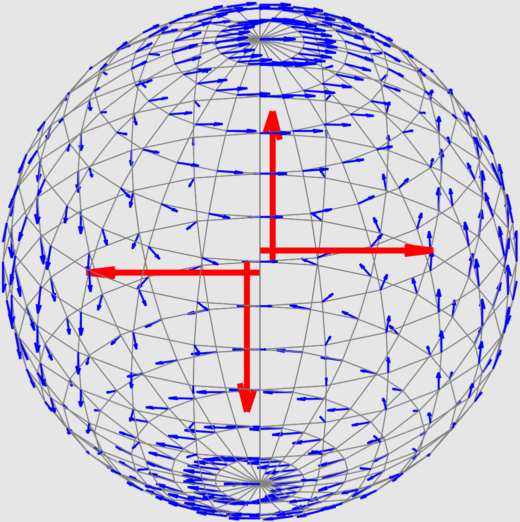
Seismic Wave Patterns

S-wave radiation pattern

$$(\mathbf{1} - \mathbf{P})\,\dot{\mathbf{M}}\vec{e}$$

for

$$\dot{\mathbf{M}} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$



Earthquake Source Theory

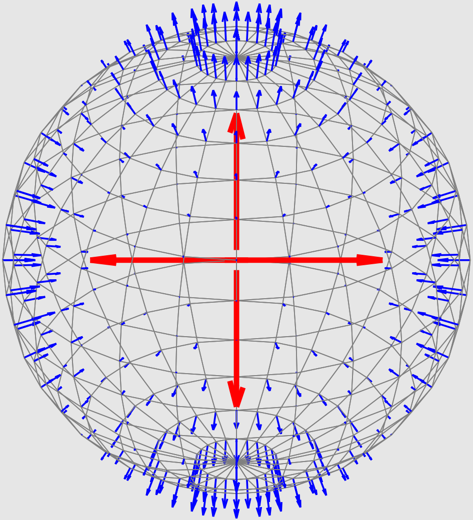
Seismic Wave Patterns

P-wave radiation pattern

$$\mathbf{P}\dot{\mathbf{M}}\vec{e}$$

for

$$\dot{\mathbf{M}} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



Earthquake Source Theory

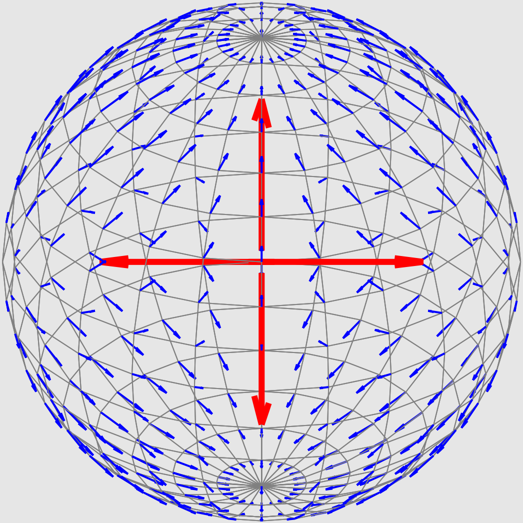
Seismic Wave Patterns

S-wave radiation pattern

$$(\mathbf{1}-\mathbf{P})\,\dot{\mathbf{M}}\vec{e}$$

for

$$\dot{\mathbf{M}} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



The Scalar Seismic Moment

If

$$\mathbf{M} = \begin{pmatrix} 0 & 0 & M_0 \\ 0 & 0 & 0 \\ M_0 & 0 & 0 \end{pmatrix} \quad \text{or} \quad \mathbf{M} = \begin{pmatrix} -M_0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & M_0 \end{pmatrix} \qquad (36)$$

(or similar), M_0 is called (scalar) seismic moment.

In general:

$$M_0 = \frac{M_1 - M_3}{2} \qquad (37)$$

where M_1, M_2, and M_3 are the eigenvalues of $\mathbf{M}$ in descending order.

Earthquake Source Theory

The Scalar Seismic Moment

Alternative interpretation of the seismic moment:

$$M_0 = \mu A \overline{u} \qquad (38)$$

where

A = size of the rupture area [m^2]
$\overline{u}$ = mean displacement along the rupture area [m]

Earthquake Source Theory

Amplitudes of Body Waves

P-wave displacement:

$$\vec{u}_p(\vec{x}, t) \; = \; \frac{s_p^3}{4\pi\rho r}\mathbf{P}\,\dot{\mathbf{M}}(t - s_p r)\vec{e} \tag{39}$$

Maximum displacement occurs in the directions of the first and third principal axes of $\dot{\mathbf{M}}$:

$$|\vec{u}_p|_{\mathrm{max}} \; = \; \frac{s_p^3}{4\pi\rho r}|\dot{M}_0|_{\mathrm{max}} \tag{40}$$

Amplitudes of Body Waves

S-wave displacement:

$$\vec{u}_s(\vec{x}, t) = \frac{s_s^3}{4\pi\rho r} \left(\mathbf{1} - \mathbf{P}\right) \dot{\mathbf{M}}(t - s_s r)\vec{e} \tag{41}$$

Maximum displacement occurs in the directions 45° between the first and third principal axis of $\dot{\mathbf{M}}$:

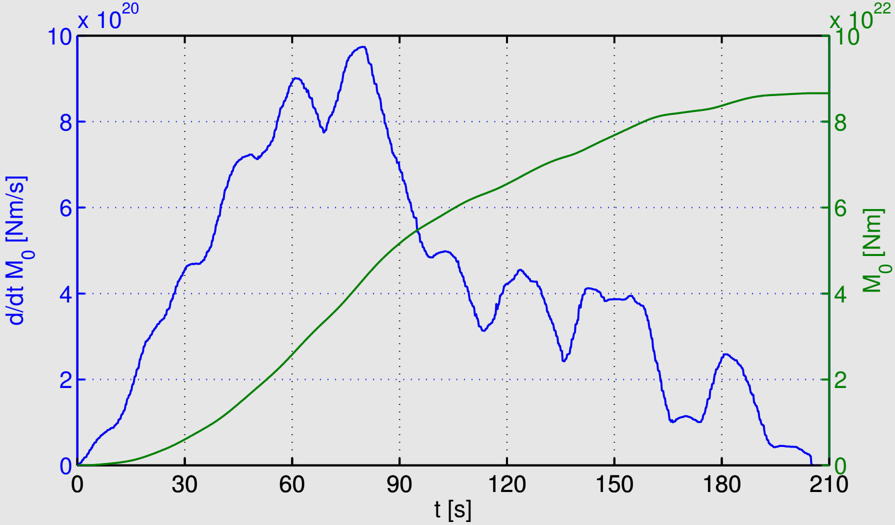
$$|\vec{u}_s|_{\max} = \frac{s_s^3}{4\pi\rho r}|\dot{M}_0|_{\max} \tag{42}$$

$$\Downarrow$$

$$\frac{|\vec{u}_s|_{\max}}{|\vec{u}_p|_{\max}} = \frac{s_s^3}{s_p^3} = \left(\frac{v_p}{v_s}\right)^3 \approx 5 \tag{43}$$

Earthquake Source Theory

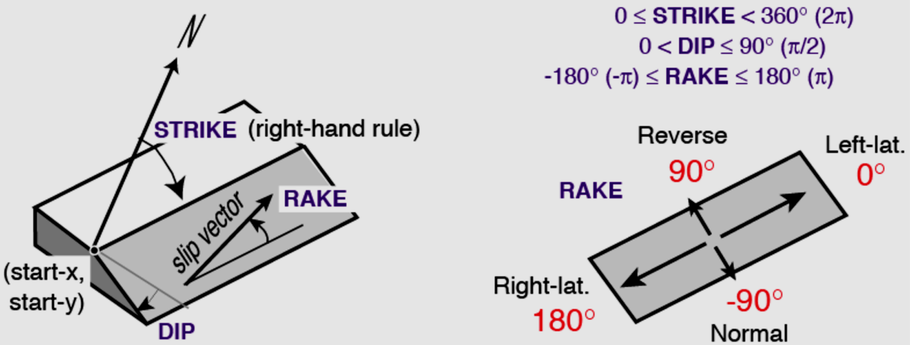
Seismic Moment vs. Moment Rate for the Alaska 1964 Earthquake



Earthquake Source Theory

The Seismic Moment Tensor for a Double Force Couple

Definition of strike ϕ, dip δ, and rake λ according to Aki and Richards (1980)



Source: Toda et al., Coulomb 3.3 User Guide

The Seismic Moment Tensor for a Double Force Couple

Step 1: Start with a force couple in x_2 direction displaced in x_3 direction:

$$\vec{F} = \begin{pmatrix} 0 \\ F \\ 0 \end{pmatrix}, \quad \vec{a} = \begin{pmatrix} 0 \\ 0 \\ a \end{pmatrix} \tag{44}$$

Step 2: Rotate $\vec{F}$ and $\vec{a}$ counterclockwise by the rake angle λ in the x_1-x_2 plane:

$$\vec{F}_\lambda = \mathbf{R}_\lambda \vec{F}, \quad \vec{a}_\lambda = \mathbf{R}_\lambda \vec{a} = \vec{a} \tag{45}$$

with

$$\mathbf{R}_\lambda = \begin{pmatrix} \cos\lambda & -\sin\lambda & 0 \\ \sin\lambda & cos\lambda & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{46}$$

Earthquake Source Theory

The Seismic Moment Tensor for a Double Force Couple

Step 3: Rotate $\vec{F}_\lambda$ and $\vec{a}_\lambda$ clockwise by the dip angle δ in the x_1-x_3 plane:

$$\vec{F}_{\lambda\delta} \;=\; \mathbf{R}_\delta \vec{F}_\lambda \;=\; \mathbf{R}_\delta \mathbf{R}_\lambda \vec{F}, \quad \vec{a}_{\lambda\delta} \;=\; \mathbf{R}_\delta \vec{a}_\lambda \;=\; \mathbf{R}_\delta \mathbf{R}_\lambda \vec{a} \qquad (47)$$

with

$$\mathbf{R}_\delta \;=\; \begin{pmatrix} \cos\delta & 0 & \sin\delta \\ 0 & 1 & 0 \\ -\sin\delta & 0 & cos\delta \end{pmatrix} \qquad (48)$$

The Seismic Moment Tensor for a Double Force Couple

Step 4: Rotate $\vec{F}_{\lambda\delta}$ and $\vec{a}_{\lambda\delta}$ clockwise by the strike angle ϕ in the x_1-x_2 plane:

$$\vec{F}_{\lambda\delta\phi} = \mathbf{R}_\phi \vec{F}_{\lambda\delta} = \mathbf{R}_\phi \mathbf{R}_\delta \mathbf{R}_\lambda \vec{F}, \quad \vec{a}_{\lambda\delta\phi} = \mathbf{R}_\phi \vec{a}_{\lambda\delta} = \mathbf{R}_\phi \mathbf{R}_\delta \mathbf{R}_\lambda \vec{a} \tag{49}$$

with

$$\mathbf{R}_\phi = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{50}$$

$$\Downarrow$$

$$\vec{F}_{\lambda\delta\phi} = \mathbf{R}\vec{F}, \quad \vec{a}_{\lambda\delta\phi} = \mathbf{R}\vec{a} \quad \text{with} \quad \mathbf{R} = \mathbf{R}_\phi \mathbf{R}_\delta \mathbf{R}_\lambda \tag{51}$$

Earthquake Source Theory

The Seismic Moment Tensor for a Double Force Couple

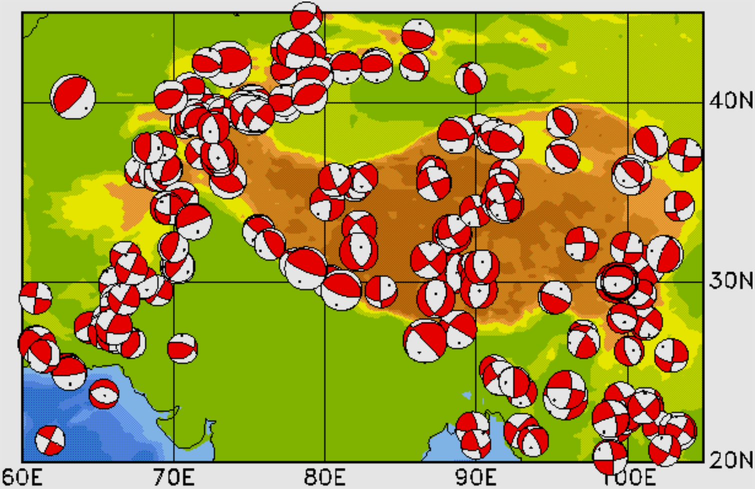
↓

$$\begin{aligned}
\mathbf{M} &= \vec{F}_{\lambda\delta\phi}\,\vec{a}^{\,T}_{\lambda\delta\phi} + \vec{a}_{\lambda\delta\phi}\,\vec{F}^{\,T}_{\lambda\delta\phi} = \left(\mathbf{R}\vec{F}\right)(\mathbf{R}\vec{a})^T + (\mathbf{R}\vec{a})\left(\mathbf{R}\vec{F}\right)^T && (52)\\
&= \mathbf{R}\left(\vec{F}\vec{a}^{\,T} + \vec{a}\vec{F}^{\,T}\right)\mathbf{R}^T && (53)\\
&= \mathbf{R}\left(\begin{pmatrix}0\\F\\0\end{pmatrix}\begin{pmatrix}0\\0\\a\end{pmatrix}^T + \begin{pmatrix}0\\0\\a\end{pmatrix}\begin{pmatrix}0\\F\\0\end{pmatrix}^T\right)\mathbf{R}^T && (54)\\
&= M_0\,\mathbf{R}\begin{pmatrix}0 & 0 & 0\\0 & 0 & 1\\0 & 1 & 0\end{pmatrix}\mathbf{R}^T && (55)
\end{aligned}$$

with $M_0 = Fa$

Earthquake Source Theory

Beachball Plots



Source: Earthquake-Report.com

Earthquake Source Theory

Beachball Plots

Basic assumption: $\dot{\mathbf{M}}(t)$ has the same shape as $\mathbf{M}(t)$,

$$\mathbf{M}(t) \;=\; f(t)\,\mathbf{M}, \quad \dot{\mathbf{M}}(t) \;=\; \dot{f}(t)\,\mathbf{M} \tag{56}$$

where $\mathbf{M}$ is the total seismic moment, and $f(t)$ increases from 0 to 1.

P-wave radiation pattern:

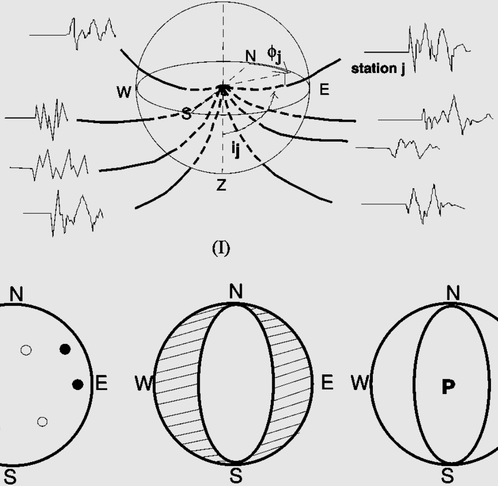
$$\vec{u} \;\propto\; \mathbf{PM}\vec{e} \;=\; ((\mathbf{M}\vec{e})\cdot\vec{e})\,\vec{e} \tag{57}$$

$$\Downarrow$$

P-wave arrives with compression first if $(\mathbf{M}\vec{e})\cdot\vec{e} > 0$ and with dilatation if $(\mathbf{M}\vec{e})\cdot\vec{e} < 0$.

Earthquake Source Theory

Beachball Plots

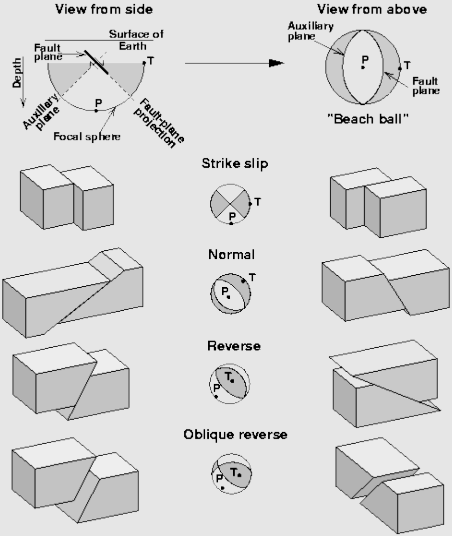


Source: Ph.D. thesis A. Belmonte-Pool, FU Berlin

Earthquake Source Theory

Beachball Plots

- Directions where the P-wave arrives first with compression $((\mathbf{M}\vec{e}) \cdot \vec{e} > 0)$ are colored.
- Directions where the P-wave arrives first with dilatation $(\mathbf{M}\vec{e}) \cdot \vec{e} < 0$ are left white.
- Projection of the lower half of the sphere is plotted (sometimes stereographic projection, but mostly equal-area projection).



Source: US Geological Survey

Earthquake Source Theory

Beachball Plots

If the eigenvalues of $\mathbf{M}$ are M_0, 0, and $-M_0$, the sphere consists of 4 equal quadrants.

Examples of beachball plots:

- normal fault ($\lambda = -90^\circ$) for different dip angles δ
- reverse fault ($\lambda = 90^\circ$) for different dip angles δ
- transform fault ($\lambda = 0^\circ$) for different dip angles δ
- fault dipping at $\delta = 45^\circ$ for different rake angles λ
- fault dipping at $\delta = 45^\circ$ for different rake angles λ with additional isotropic expansion

Earthquake Magnitude

Intensity and Magnitude

Intensity describes the severity of an earthquake in terms of its effects on the Earth's surface and on humans and their structures.

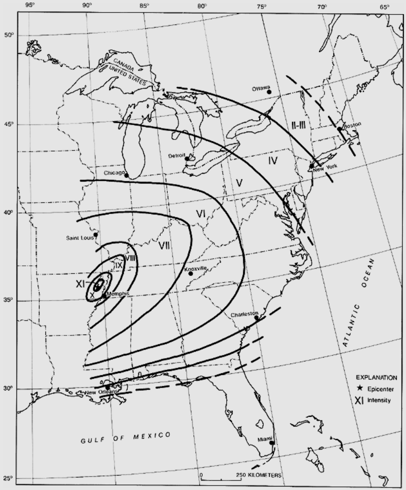
- Usually written as a Roman numeral.
- Goes back to a 12 level scale (originally 10) from I (not felt) to XII (total destruction) named after G. Mercalli (1850–1914).
- Several extensions / refinements: MCS (Mercalli-Cancani-Sieberg) scale, MWN (Mercalli-Wood-Neumann) scale, MSK scale (Medvedev, Sponheuer & Karnik, 1964), EMS-98 scale (European Macroseismic Scale, 2000).

Magnitude characterizes the size of an earthquake using measured values.

- Usually written as an Arabic numeral with one decimal digit.
- Several different magnitude definitions.
- Logarithmic scale.

Earthquake Magnitude

Example of an Isoseismal Map of Earthquake Intensity



Source: USGS

Earthquake Magnitude

General Definition of Earthquake Magnitude

If X is any physically measured property of an earthquake, e. g.

- total seismic moment M_0 or
- maximum ground displacement $|\vec{u}|_{\mathrm{max}}$,

the corresponding earthquake magnitude is defined by

$$M_X \;=\; e \log_{10}\left(\frac{X}{X_0}\right) \tag{58}$$

where

X_0 = measured value for an earthquake of $M_X = 0$ under the same conditions

e = factor used for making different magnitude definitions consistent (mostly $e = 1$)

Earthquake Magnitude

General Definition of Earthquake Magnitude

If X is a property related to any point different from the earthquake focus, X_0 is a function of distance Δ and depth h (and other properties).

$$\Downarrow$$

$$M_X \;=\; e\,\log_{10}\left(\frac{X}{X_0(\Delta, h)}\right) \;=\; e\,\log_{10} X + \sigma(\Delta, h) \tag{59}$$

with the distance-depth correction function

$$\sigma(\Delta, h) \;=\; -\,e\,\log_{10} X_0(\Delta, h) \tag{60}$$

This only makes sense if the distance-depth dependence of X is independent of X itself.

Earthquake Magnitude

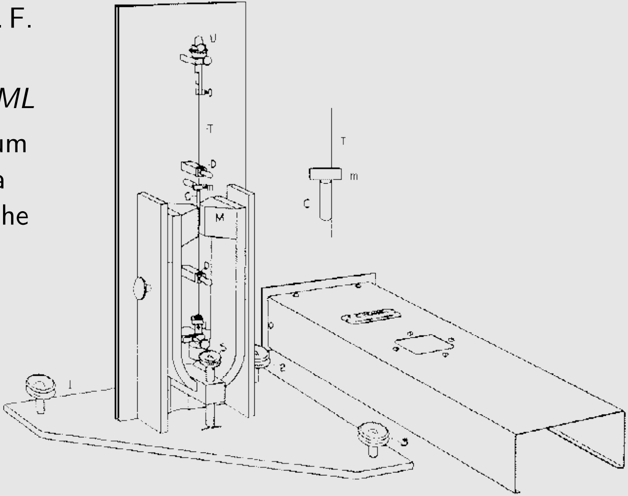
Upper und Lower Limits of Magnitude Scales

- All magnitude scales are from their definition open and both ends.
- Upper limits on Earth are introduced by geological constraints and by the process of wave propagation.
- Negative magnitudes are possible. The definition of zero magnitude is arbitrary and corresponds to what was detectable when the first magnitude definition (C. F. Richter, 1935) was introduced.

Earthquake Magnitude

The Local Magnitude (Richter Scale)

- Introduced by C. F. Richter in 1935.
- Symbol: M_L or ML
- X is the maximum amplitude A of a specific device, the Wood-Anderson seismometer.



Source: Southern California Earthquake Data Center

Earthquake Magnitude

The Wood-Anderson Seismometer

- Oscillation by torsion of a wire
- Electromagnetic damping
- Natural period of $\approx 0.8\,\mathrm{s}$ (frequency $f_0 = 1.25\,\mathrm{Hz}$); close to the natural period of many building structures.

$$\Downarrow$$

Relevant for earthquake hazard.

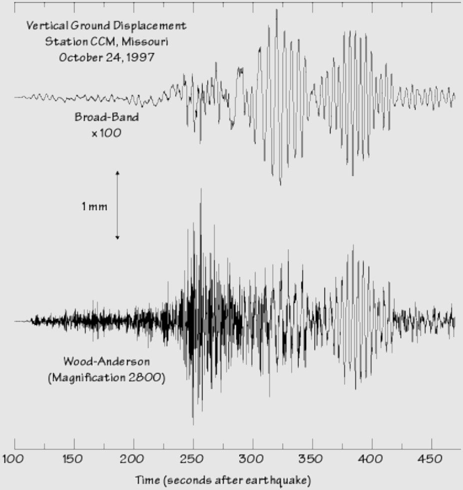
- Maximum magnification (record vs. ground displacement) of ≈ 2080 at f_0; sometimes a wrong value of 2800 was assumed.

$$\Downarrow$$

Local magnitudes derived from synthesized seismograms were too high for some time.

Earthquake Magnitude

The Wood-Anderson Seismometer



Source: C. J. Ammon, Pennsylvania State University

Earthquake Magnitude

The Local Magnitude (Richter Scale)

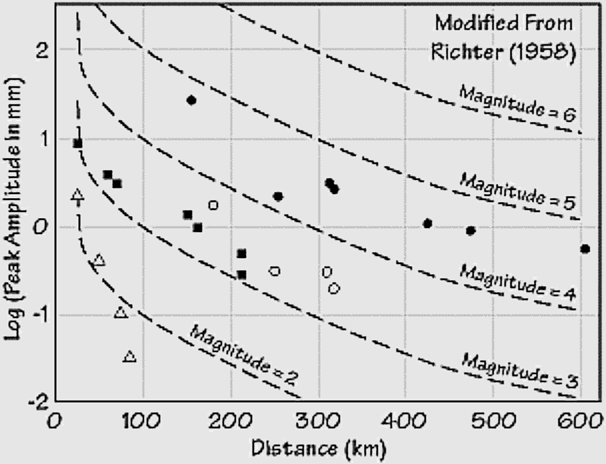
- The local magnitude was originally defined as

$$M_L \;=\; \log_{10} A \tag{61}$$

  where the maximum amplitude A of the Wood-Anderson seismometer is measured in μm at 100 km distance from the epicenter.
- $e = 1$ → 1 unit increase in magnitude corresponds to an increase in the instrument's amplitude by a factor 10.
- Originally only a distance correction $\sigma(\Delta) = -\log_{10} A_0(\Delta)$ for shallow earthquakes ($h \leq 15\,\mathrm{km}$) in California was provided.

Earthquake Magnitude

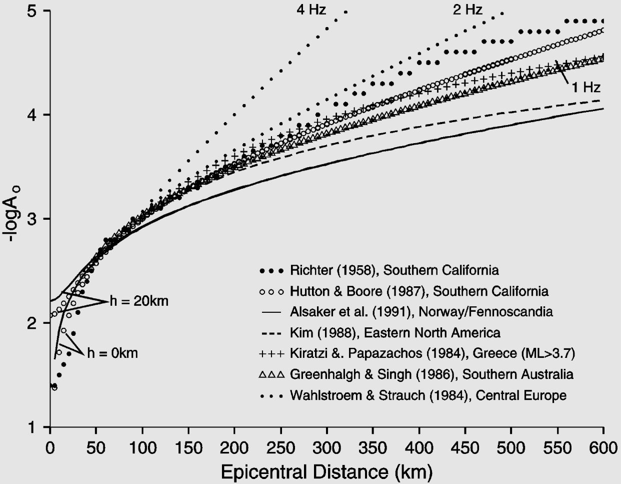
Richter's Original Distance Correction



Source: C. J. Ammon, Pennsylvania State University

Earthquake Magnitude

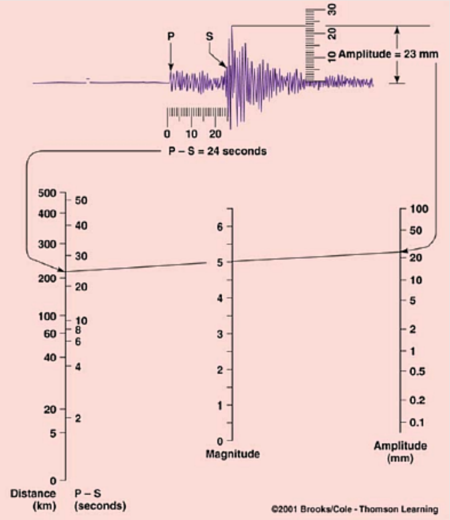
Distance Corrections for Different Regions and Depths



Source: Bormann (ed), New Manual of Seismological Observatory Practice

Intensity and Magnitude

Determining the Local Magnitude of an Earthquake



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Earthquake Magnitude

The Surface-Wave Magnitude

- Symbol: M_S or MS
- Original definition by B. Gutenberg (1945):

$$M_S \;=\; \log_{10} u_{h\,\max} + \sigma(\Delta) \tag{62}$$

where $u_{h\,\max}$ is the maximum horizontal ground displacement at periods from $T = 18\,\mathrm{s}$ to $22\,\mathrm{s}$.

- Widely used modified definition (Moscow-Prague formula, 1962):

$$M_S \;=\; \max\left\{\log_{10} \frac{|\vec{u}|}{T}\right\} + 1.66 \log_{10} \Delta + 3.3 \tag{63}$$

for $2^\circ \leq \Delta \leq 160^\circ$. The maximum is taken over all periods of surface waves.

Earthquake Magnitude

Body-Wave Magnitudes

- Two significantly different definitions
- Symbols: m_B, mB, m_b, mb,
- Original definition by B. Gutenberg (1945):

$$m_B \;=\; \max\left\{\log_{10}\frac{|\vec{u}|}{T}\right\}+\sigma(\Delta) \tag{64}$$

  where $|\vec{u}|$ is analyzed for different types of body waves separately (with different functions $\sigma(\Delta)$ at periods from $T = 0.5\,\mathrm{s}$ to $12\,\mathrm{s}$.
- Alternative definition (m_b, mb) refers to higher-frequency components of P-waves only.

Earthquake Magnitude

The Moment Magnitude

$$M_W \;=\; \tfrac{2}{3}\log_{10} M_0 - 6.1 \tag{65}$$

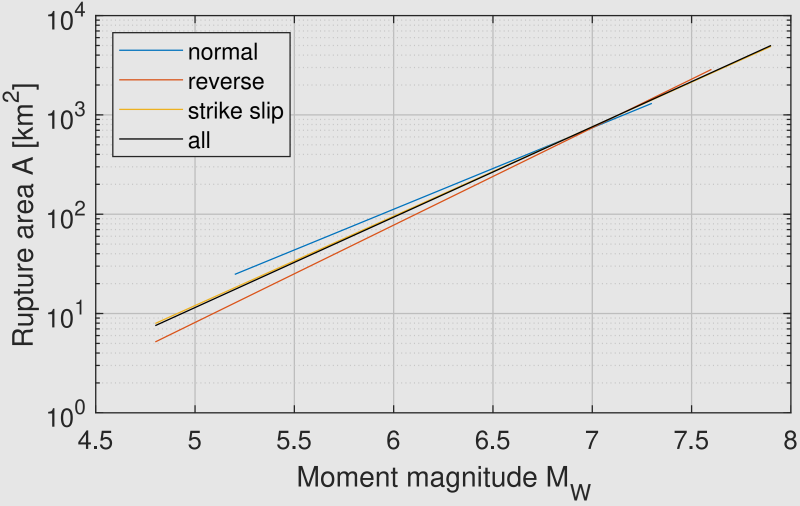
with M_0 in Nm

- Introduced in 1977 by H. Kanamori in order to characterize large earthquakes.
- More closely related to the strength of earthquakes at the seismic focus than older magnitude definitions.
- Rather a tectonic than a seismological magnitude scale.

Why is $e = \frac{2}{3}$ here while $e = 1$ for other magnitude scales?

Earthquake Magnitude

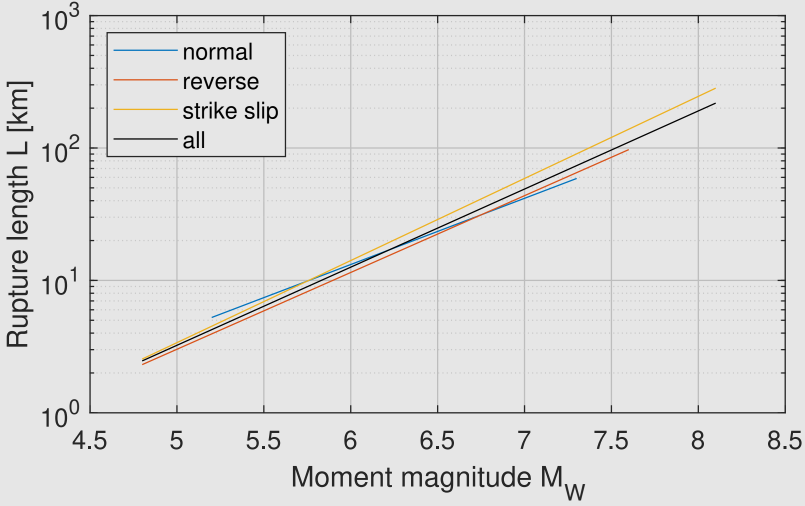
Scaling Properties of Earthquakes



Data: Wells & Coppersmith, Bull. Seismol. Soc. Am., 1994

Earthquake Magnitude

Scaling Properties of Earthquakes



Data: Wells & Coppersmith, Bull. Seismol. Soc. Am., 1994

Earthquake Magnitude

Scaling Properties of Earthquakes

Assume that the forces F in M_0 were uniformly distributed over an area A.

Shear stress

$$\sigma_s = \frac{F}{A} \tag{66}$$

↓

Shear strain

$$\gamma = \frac{\sigma_s}{\mu} = \frac{F}{\mu A} \tag{67}$$

↓

Displacement in shear direction

$$u = \gamma a = \frac{Fa}{\mu A} = \frac{M_0}{\mu A} \tag{68}$$

Earthquake Magnitude

Scaling Properties of Earthquakes

Alternative interpretation of the seismic moment:

$$M_0 = \mu \, A \, \overline{u} \tag{69}$$

where

$$\begin{aligned} A &= \text{size of the rupture area [m}^2\text{]} \\ \overline{u} &= \text{mean displacement along the rupture area [m]} \end{aligned}$$

Relation between A und $\overline{u}$?

Earthquake Magnitude

Scaling Properties of Earthquakes

Assume an elastic solid in a domain with a characteristic length scale L. Increase L, but keep displacement $\vec{u}$.

$$\Downarrow$$

$$\boldsymbol{\epsilon} \; = \; \frac{1}{2}\left(\nabla\vec{u} + (\nabla\vec{u})^T\right) \; \sim \; \frac{1}{L} \tag{70}$$

$$\Downarrow$$

$$\boldsymbol{\sigma} \; = \; \lambda\,\epsilon_v\,\mathbf{1} + 2\mu\,\boldsymbol{\epsilon} \; \sim \; \frac{1}{L} \tag{71}$$

Earthquake Magnitude

Scaling Properties of Earthquakes

$\boldsymbol{\sigma}$ remains constant if $\vec{u}$ is also upscaled in the way $\vec{u} \sim L$.

⬇

$$\overline{u} \;\sim\; \sqrt{A} \tag{72}$$

if

- the stress drop (the component resulting from the forces in M_0) of all earthquakes is the same and
- the rupture areas of all earthquakes are geometrically similar.

⬇

$$\overline{u} \;\sim\; M_0^{\frac{1}{3}}, \quad A \;\sim\; M_0^{\frac{2}{3}} \tag{73}$$

Earthquake Magnitude

Scaling Properties of Earthquakes

Rupture propagates along the rupture area at a given velocity

$$\Downarrow$$

$$\text{Duration} \quad \tau \sim \sqrt{A} \sim M_0^{\frac{1}{3}} \tag{74}$$

$$\Downarrow$$

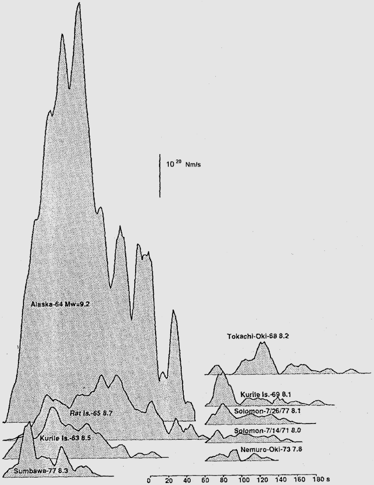
$$|\vec{u}| \sim \dot{M}_0 \sim \frac{M_0}{\tau} \sim M_0^{\frac{2}{3}} \tag{75}$$

$$\Downarrow$$

Magnitude definition based on M_0 requires $e = \frac{2}{3}$.

Earthquake Magnitude

Scaling Properties of Earthquakes



Source: Bormann (ed), New Manual of Seismological Observatory Practice

Earthquake Magnitude

The Energy Magnitude

A crude scaling relation: particle velocity

$$|\vec{v}| \sim \frac{|\vec{u}|}{\tau} \sim \frac{M_0^{\frac{2}{3}}}{\tau} \sim M_0^{\frac{1}{3}} \tag{76}$$

$$\Downarrow$$

Total radiated kinetic energy

$$E_{\text{kin}} \sim |\vec{v}|^2\tau \sim M_0 \tag{77}$$

Potential energy equals kinetic energy in the mean.

$$\Downarrow$$

Total radiated seismic energy

$$E \sim M_0 \tag{78}$$

Earthquake Magnitude

The Energy Magnitude

Theoretical relationship suggested by H. Kanamori (1977):

$$E \approx 5 \times 10^{-5} M_0 \tag{79}$$

Corresponding definition of the energy magnitude:

$$M_E = \tfrac{2}{3} \log_{10} \frac{E}{5 \times 10^{-5}} - 6.1 = \tfrac{2}{3} \log_{10} E - 3.2 \tag{80}$$

Up to one order of magnitude deviation from Kanamori's relationship was found for individual earthquakes.

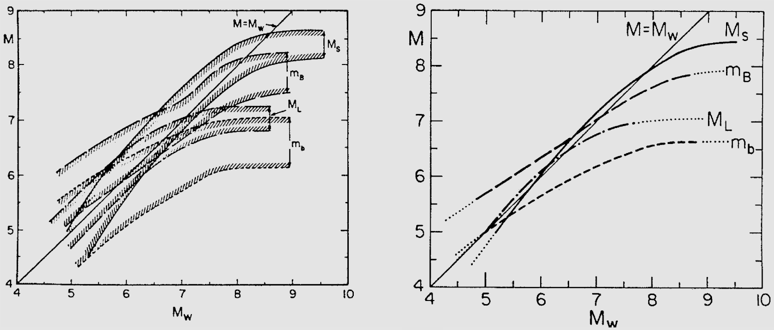
↓

Significant differences between M_E and M_W for individual earthquakes.

Earthquake Magnitude

Saturation of Magnitudes

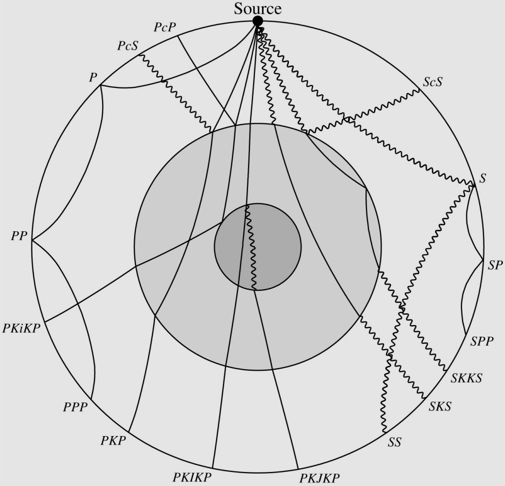
- All magnitudes based on recording seismic waves fall below M_W for large earthquakes.
- Effect is stronger if short-term (high-frequency) components of the seismic waves are used.



Source: Bormann (ed), New Manual of Seismological Observatory Practice

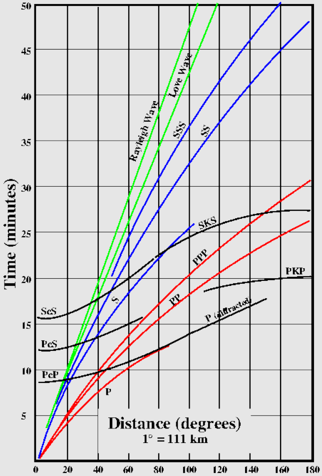
Propagation of Seismic Waves in Inhomogeneous Media

Global Wave Propagation in the Earth's Interior



Source: Shearer, Introduction to Seismology

Travel Time Curves



Source: Southern Arizona Seismic Observatory

Propagation of Seismic Waves in Inhomogeneous Media

Ray and Wave Front Approaches

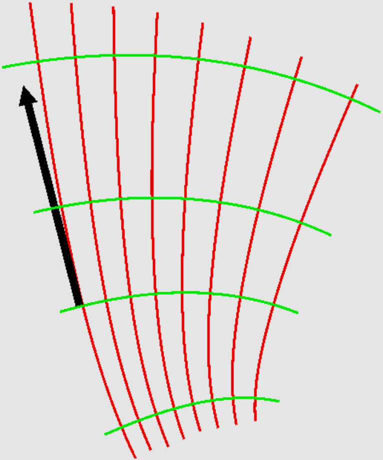
Plane wave approach and spherical wave solutions are only valid in a homogeneous medium.

Generalization:

Wave fronts are no longer planes, but arbitrary surfaces.

Direction of propagation is normal to the wave fronts

Ray paths are lines following the local direction of propagation (no longer straight lines).



Source: Wikipedia, © ???

Propagation of Seismic Waves in Inhomogeneous Media

Ray and Wave Front Approaches

Extensions towards the plane wave approach:

Replace

$$\vec{u}(\vec{x}, t) \;=\; f(t - \vec{s} \cdot \vec{x})\, \vec{a} \tag{81}$$

by

$$\vec{u}(\vec{x}, t) \;=\; f(t - \psi(\vec{x}))\, \vec{a}(\vec{x}) \tag{82}$$

- Retarded time $\tau = t - \psi(\vec{x})$ instead of $\tau = t - \vec{s} \cdot \vec{x}$ with a general phase function $\psi(\vec{x})$
- Spatially variable amplitude vector $\vec{a}(\vec{x})$

Propagation of Seismic Waves in Inhomogeneous Media

Ray and Wave Front Approaches

Wave fronts: surfaces where $\psi(\vec{x})$ is constant

Direction of propagation: Wave propagates locally in direction of $\nabla\psi(\vec{x})$.

Slowness: $\vec{s}(\vec{x}) = \nabla\psi(\vec{x})$ defines the local slowness vector.

Ray paths: lines following the direction of $\vec{s} = \nabla\psi$

Limitation: Only valid in the limit of high frequencies.

Propagation of Seismic Waves in Inhomogeneous Media

Ray and Wave Front Approaches

Main results:

Types of waves: two independent types of waves (P-wave and S-wave); amplitude vector $\vec{a}$ is either parallel or normal to $\vec{s} = \nabla\psi$ at each point; no reflection; no merging of both wave types

Slowness/velocity: same relationship as for plane and spherical waves (eikonal equation)

$$|\vec{s}|^2 \;=\; |\nabla\psi|^2 \;=\; \frac{\rho}{\lambda + 2\mu} \qquad \text{or} \qquad |\vec{s}|^2 \;=\; |\nabla\psi|^2 \;=\; \frac{\rho}{\mu} \tag{83}$$

Ray and Wave Front Approaches

Amplitudes: Absolute values of the amplitudes are determined by the condition

$$\mathrm{div}(\vec{q}) \; = \; 0 \tag{84}$$

with the flux density

$$\vec{q} \; = \; \rho\,|\vec{a}|^2\,\vec{v} \; = \; \epsilon\,\vec{v} \tag{85}$$

and $\epsilon = \rho|\vec{a}|^2$.

ϵ is proportional to the energy density (elastic energy + kinetic energy per volume) of a harmonic wave.

⬇

Call ϵ energy density and $\vec{q}$ energy flux density.

Propagation of Seismic Waves in Inhomogeneous Media

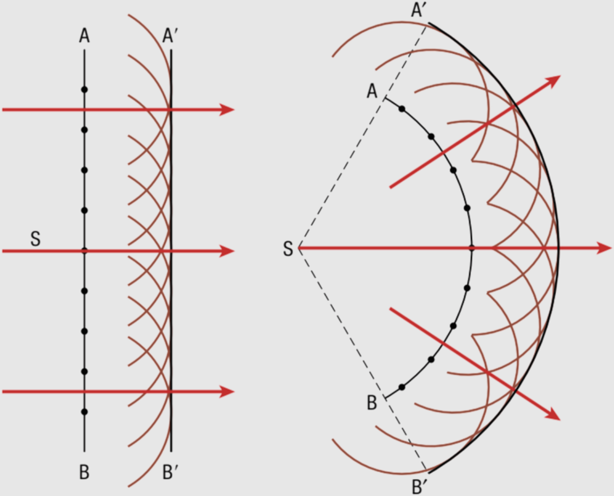
Geometrical and Numerical Approaches

Numerical solution of the eikonal equation: one-sided difference quotients for $\psi(\vec{x})$ from the considered points to neighbors where ψ is already known; similar to upstream scheme.

Huygens' princple: Construct wave fronts (planes where $\psi(\vec{x})$ is constant) progressively for increasing values of ψ; rather a geometrical approach than a numerical method.

Ray tracing: Compute individual rays (lines in dierection of $\nabla\psi(\vec{x})$).

Construction of Wave Fronts – Huygens' Princple



Source: WillowWood Lessons

Propagation of Seismic Waves in Inhomogeneous Media

Construction of Ray Paths

Each ray path $\vec{r}(t)$ must be normal to the wave fronts ($\phi = \text{const}$).

$$\Downarrow$$

$$\frac{d}{dt}\vec{r}(t) \;\sim\; \nabla\psi(\vec{r}(t)) \;=\; \vec{s}(\vec{r}(t)) \tag{86}$$

Follow the slowness vector along the ray path:

$$\begin{aligned}
\frac{d}{dt}\vec{s}(\vec{r}(t)) &= \nabla\vec{s}(\vec{r}(t))\,\frac{d}{dt}\vec{r}(t) && (87)\\
&\sim \nabla\vec{s}(\vec{r}(t))\,\vec{s}(\vec{r}(t)) && (88)\\
&= \frac{1}{2}\nabla|\vec{s}(\vec{r}(t))|^2 \quad \text{because} \quad \nabla\vec{s}^{\,T} = \nabla\vec{s} && (89)\\
&= |\vec{s}(\vec{r}(t))|\,\nabla|\vec{s}(\vec{r}(t))| && (90)
\end{aligned}$$

Propagation of Seismic Waves in Inhomogeneous Media

Construction of Ray Paths

$\vec{s}$ changes along the ray path in direction of increasing slowness (decreasing velocity).

$$\Downarrow$$

Component of $\vec{s}$ parallel to the surfaces of constant slowness remains constant.

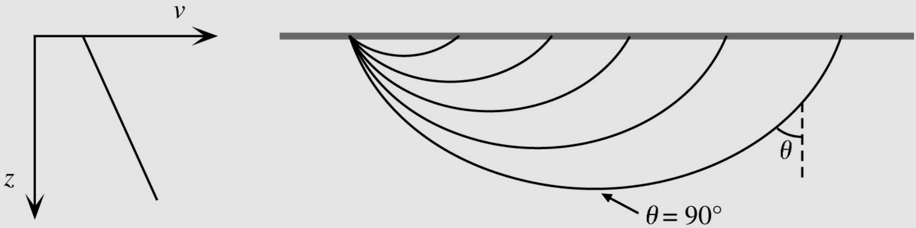
$$\Downarrow$$

Horizontal slowness remains constant.

Horizontal direction in seismology = planes where the properties of the material are constant.

Propagation of Seismic Waves in Inhomogeneous Media

Example: Velocity Continuously Increasing with Depth

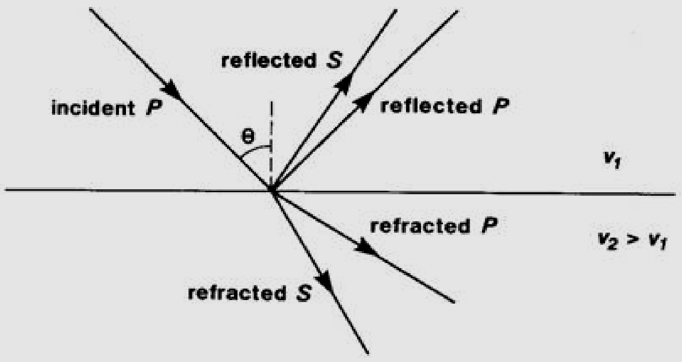


Source: Shearer, Introduction to Seismology

Propagation of Seismic Waves in Inhomogeneous Media

Reflection and Refraction

Simplest case: two homogeneous, isotropic halfspaces with different properties (λ, μ, ρ) and plane waves in each of them; horizontal interface at $x_3 = 0$.



Source: University College London

Propagation of Seismic Waves in Inhomogeneous Media

Reflection and Refraction

Each wave is considered as a plane wave

$$\vec{u}(\vec{x}, t) \;=\; f(t - \vec{s} \cdot \vec{x})\, \vec{a} \tag{91}$$

with individual values $\vec{s}$ and $\vec{a}$ where either

$$\vec{a} \;\|\; \vec{s} \qquad \text{and} \qquad |\vec{s}| \;=\; s_p \;=\; \sqrt{\frac{\rho}{\lambda + 2\mu}} \tag{92}$$

or

$$\vec{a} \;\perp\; \vec{s} \qquad \text{and} \qquad |\vec{s}| \;=\; s_s \;=\; \sqrt{\frac{\rho}{\mu}} \tag{93}$$

Reflection and Refraction

Displacement must be continuous at $x_3 = 0$.

$$\Downarrow$$

All waves must have the same retarded time

$$\tau \;=\; t - \vec{s} \cdot \vec{x} \;=\; t - s_1 x_1 - s_2 x_2 \tag{94}$$

at the interface.

$$\Downarrow$$

Components of $\vec{s}$ parallel to the interface $(s_1,\, s_2)$ must be the same for all waves.

$$\Downarrow$$

Horizontal slowness remains constant in reflection and refraction.

Propagation of Seismic Waves in Inhomogeneous Media

Reflection and Refraction

Second condition at the interface:

Stress acting on the interface, given by

$$\vec{\sigma}_{\text{int}} \;=\; \boldsymbol{\sigma}\big|_{x_3=0} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{95}$$

must be continuous.

Continuity of displacement and stress at the interface

$$\Downarrow$$

Linear equation system for the amplitude vectors $\vec{a}$ of the five involved waves.

Propagation of Seismic Waves in Inhomogeneous Media

Conversion of Waves in Reflection and Refraction

Align the coordinate system in such a way that all waves propagate in the x_1-x_3 plane ($s_2 = 0$, possible because s_1 and s_2 are the same for all involved waves).

↓

Vertically polarized S-wave (SV-wave):

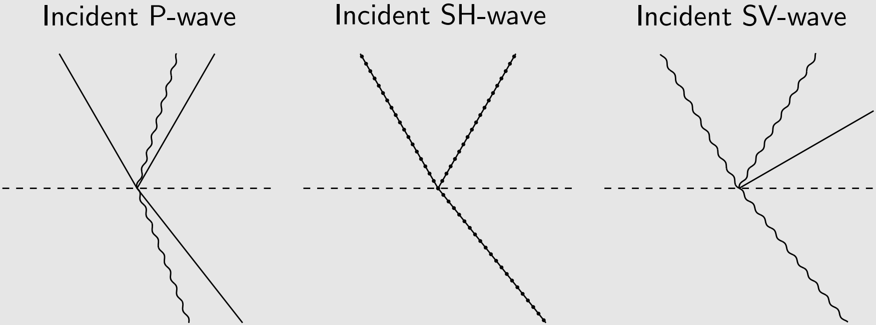
- $a_2 = 0$ → particle displacement in the x_1-x_3 plane (and normal to wave propagation)
- Converted to (and from) P and SV waves in reflection and refraction

Horizontally polarized S-wave (SH-wave):

- $a_1 = a_3 = 0$ → particle displacement in x_2 direction
- Independent of P and SV waves

Propagation of Seismic Waves in Inhomogeneous Media

Conversion of Waves in Reflection and Refraction



Interface Waves and Surface Waves

The Waves Disappearing in Reflection and Refraction

For all waves involved in refection and refraction at a planar interface ($s_2 = 0$): s_1 given and

$$|\vec{s}|^2 \;=\; s_1^2 + s_3^2 \;=\; \left\{ \begin{array}{rcl} \frac{1}{v_p^2} &=& \frac{\rho}{\lambda+2\mu} \\ \frac{1}{v_s^2} &=& \frac{\rho}{\mu} \end{array} \right. \quad \text{for} \quad \begin{array}{l} \text{P-waves} \\ \text{S-waves} \end{array} \tag{96}$$

given.

$$\Downarrow$$

Waves cannot propagate if $s_1 > |\vec{s}|$, but

$$s_3 \;=\; \pm\sqrt{|\vec{s}|^2 - s_1^2} \;=\; \pm\, i\sqrt{s_1^2 - |\vec{s}|^2} \tag{97}$$

would be a formal solution.

Interface Waves and Surface Waves

Harmonic Interface Waves

For a harmonic wave:

$$\vec{u}(\vec{x},t) \;=\; e^{i\omega(t-\vec{s}\cdot\vec{x})}\,\vec{a} \;=\; e^{i\omega\left(t-s_1x_1\mp i\sqrt{s_1^2-|\vec{s}|^2}x_3\right)}\,\vec{a} \tag{98}$$

$$\;=\; e^{i\omega(t-s_1x_1)}\,e^{\pm\omega\sqrt{s_1^2-|\vec{s}|^2}x_3}\,\vec{a} \tag{99}$$

Can be considered as a wave propagating along the interface (here in x_1 direction) with an amplitude depending on x_3:

$$\vec{u}(\vec{x},t) \;=\; e^{i\omega(t-s_1x_1)}\,\vec{a}_{\mathrm{eff}} \tag{100}$$

with the effective amplitude

$$\vec{a}_{\mathrm{eff}} \;=\; e^{\pm\omega\sqrt{s_1^2-|\vec{s}|^2}x_3}\,\vec{a} \tag{101}$$

Interface Waves and Surface Waves

Harmonic Interface Waves

Only the version where $\vec{a}_{\text{eff}}$ decreases exponentially with distance from the interface ($|x_3|$) makes sense:

$$\vec{a}_{\text{eff}} \; = \; e^{-\omega\sqrt{s_1^2-|\vec{s}|^2}|x_3|}\,\vec{a} \tag{102}$$

Respective waves are called interface waves.

Interface Waves and Surface Waves

The Depth of Penetration of Interface Waves

$$\vec{a}_{\text{eff}} \;=\; e^{-\omega\sqrt{s_1^2-|\vec{s}|^2}|x_3|}\,\vec{a} \;=\; e^{-\frac{|x_3|}{d}}\,\vec{a} \tag{103}$$

with the depth of penetration

$$d \;=\; \frac{1}{\omega\sqrt{s_1^2-|\vec{s}|^2}} \;=\; \frac{L}{2\pi\sqrt{1-\frac{|\vec{s}|^2}{s_1^2}}} \tag{104}$$

and the wavelength $L = \frac{2\pi}{\omega s_1}$.

- $d \to \infty$ (plane wave propagating along the surface) if the wave is only slightly too slow for the medium ($s_1 \to |\vec{s}|$).
- $d \to \frac{L}{2\pi}$ if the wave is much too slow for the medium ($s_1 \gg |\vec{s}|$).

Interface Waves and Surface Waves

Particle Orbits of P and SV Interface Waves

Examples of particle orbits for an incident SV wave at the crust-mantle boundary: $\alpha = 20°$, $\alpha = 30°$, $\alpha = 40°$, $\alpha = 70°$

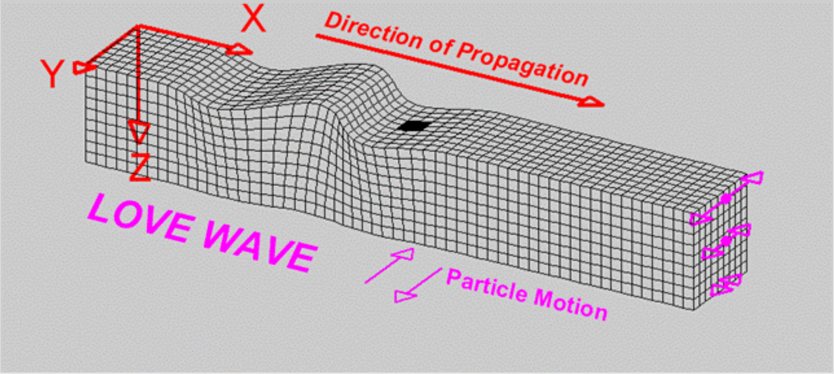
General properties:

- Particles move on elliptical orbits.
- Horizontal ellipses for P interface waves, vertical ellipses for S interface waves.
- Aspect ratio of the ellipses $= \frac{1}{\sqrt{1-\frac{|\vec{s}|^2}{s_1^2}}}$.
- Prograde rotation in the lower halfspace; retrograde rotation in the upper halfspace.

Interface Waves and Surface Waves

Particle Orbits of SH Interface Waves

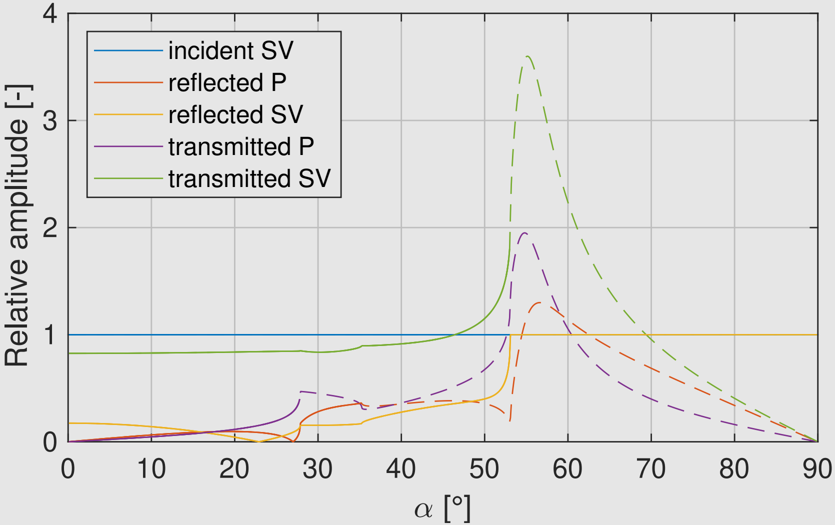
Particles move on straight lines parallel to the interface (same as for regular SH waves).



Source: L. Braile, Purdue University

Interface Waves and Surface Waves

Amplitudes at the Crust-Mantle Boundary



Interface Waves and Surface Waves

Surface Waves at a Free Surface

Interface waves: driven by plane waves (incident, reflected, refracted)

Surface waves: living on their own at a free surface (interface to air);
$\vec{\sigma}_{\text{surf}} = \vec{0}$

$\Downarrow$

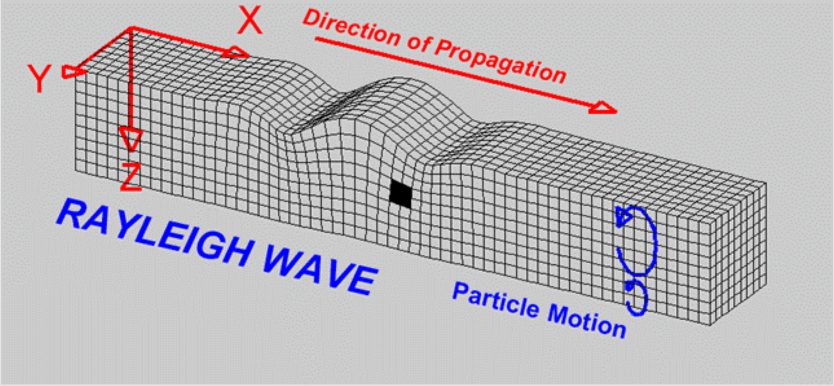
Two fundamental types of surface waves in a semi-infinite halfspace:

Rayleigh wave, named after J. W. Strutt (later 3. Lord Rayleigh)

Love wave, named after A. E. H. Love
Not possible in a homogeneous halfspace because $\vec{\sigma}_{\text{surf}} \neq \vec{0}$ for $a_2 \neq 0$

Interface Waves and Surface Waves

The Rayleigh Wave

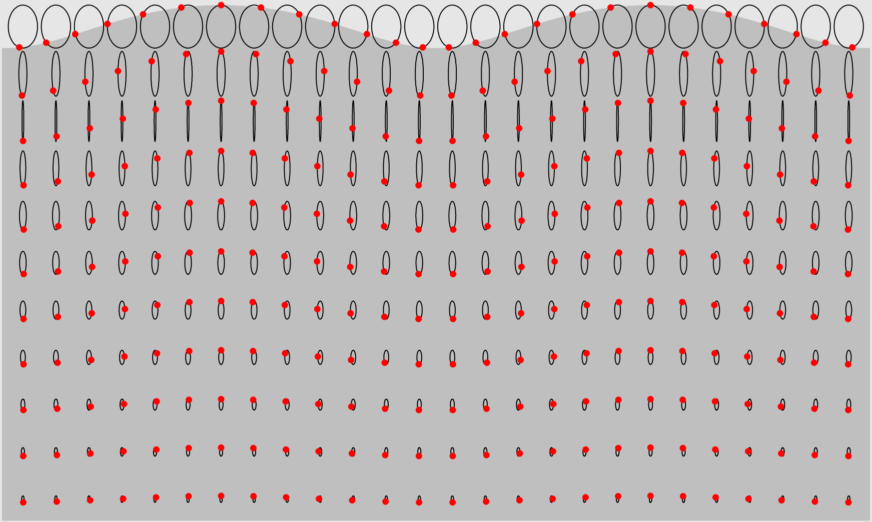


Source: L. Braile, Purdue University

Specific superposition of P and SH interface wave so that $\vec{\sigma}_{\text{surf}} = \vec{0}$

Interface Waves and Surface Waves

The Rayleigh Wave in a Homogeneous Poisson Solid ($\lambda = \mu$)

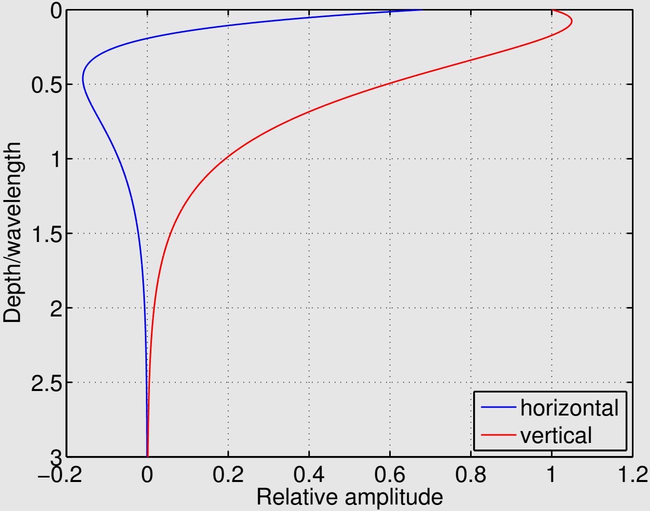


Interface Waves and Surface Waves

The Rayleigh Wave in a Homogeneous Poisson Solid ($\lambda = \mu$)

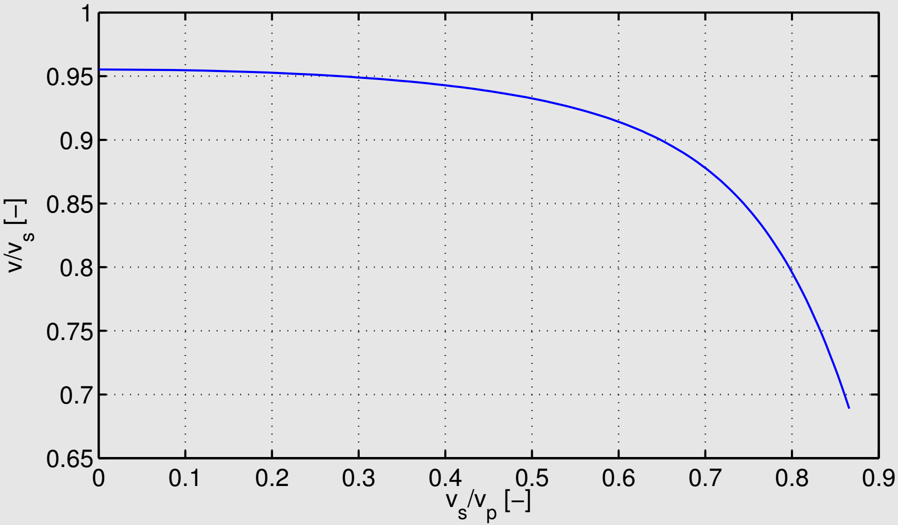
- Retrograde particle motion on elliptical orbits at the surface.
- Prograde particle motion on elliptical orbits at greater depth.
- Velocity $v \approx 0.92\, v_s$.

The Rayleigh Wave in a Homogeneous Poisson Solid ($\lambda = \mu$)



Interface Waves and Surface Waves

The Rayleigh Wave in a Homogeneous Halfspace



Interface Waves and Surface Waves

Surface Waves in Inhomogeneous Media

Assume that ρ, λ, and μ depend on the vertical coordinate x_3, and generalize

$$\vec{u}(\vec{x},t) = e^{i\omega(t-s_1x_1)}\left(e^{k_1S_px_3}a_p\begin{pmatrix}1\\0\\iS_p\end{pmatrix}+e^{k_1S_sx_3}a_s\begin{pmatrix}1\\0\\\frac{i}{S_s}\end{pmatrix}\right) \quad (105)$$

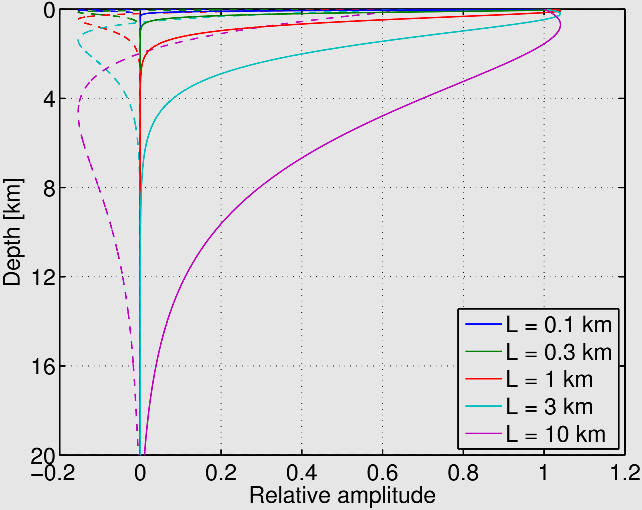
to

$$\vec{u}(\vec{x},t) = e^{i\omega(t-s_1x_1)}\,\vec{a}(x_3) \quad (106)$$

$$\Downarrow$$

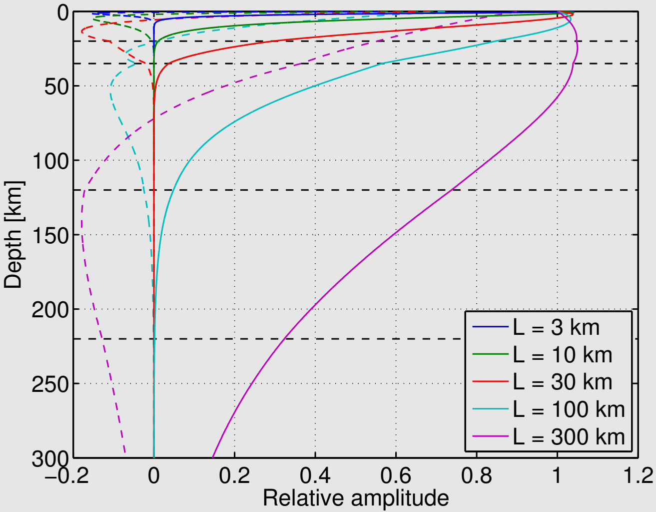
- Differential equations (eigenvalue problem) for $a_1(x_3)$, $a_2(x_3)$, $a_3(x_3)$.
- $a_1(x_3)$ and $a_3(x_3)$ are coupled (➔ Rayleigh wave), $a_2(x_3)$ is independent of $a_1(x_3)$ and $a_3(x_3)$ (➔ Love wave).
- Must be solved numerically.

Rayleigh Waves in Typical Continental Subsurface (PEM)

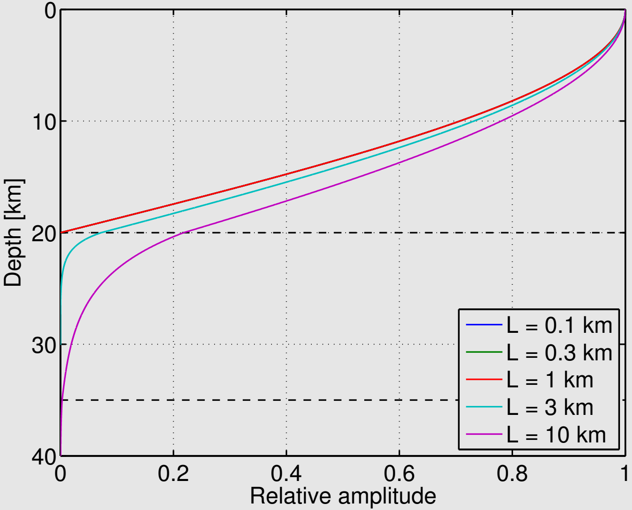


Interface Waves and Surface Waves

Rayleigh Waves in Typical Continental Subsurface (PEM)

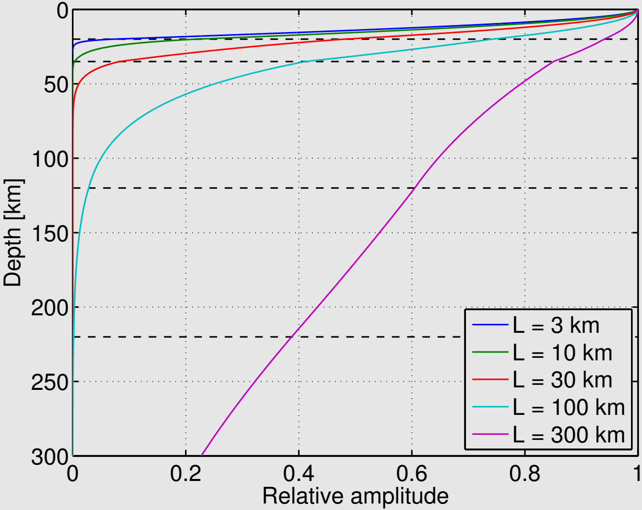


Love Waves in Typical Continental Subsurface (PEM)



Interface Waves and Surface Waves

Love Waves in Typical Continental Subsurface (PEM)



Interface Waves and Surface Waves

Main Differences Between Body Waves and Surface Waves

- Decrease with the distance from the hypocenter/epicenter r:

	Energy flux density	Amplitude
body waves	$\propto \frac{1}{r^2}$	$\propto \frac{1}{r}$
surface waves	$\propto \frac{1}{r}$	$\propto \frac{1}{\sqrt{r}}$

$$\Downarrow$$

Surface waves have a longer range than body waves.

- Velocity of harmonic surface waves depends on the wavelength (dispersion).

Interface Waves and Surface Waves

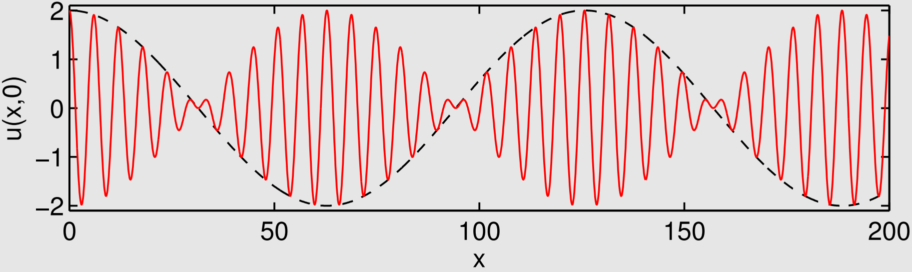
Dispersion

Simplest situation: superposition of two harmonic waves with the same amplitude (= 1), but different frequencies in 1D:

$$u(x,t) \;=\; e^{i\omega_1(t-s_1x)} + e^{i\omega_2(t-s_2x)} \;=\; e^{i(\omega_1 t-k_1x)} + e^{i(\omega_2 t-k_2x)} \qquad (107)$$

with $k_1 = \omega_1 s_1$, $k_2 = \omega_2 s_2$.

Example: $k_1 = 10$, $k_2 = 11$



Interface Waves and Surface Waves

Dispersion

$$\begin{align}
u(x,t) &= e^{i\omega_1(t-s_1x)} + e^{i\omega_2(t-s_2x)} = e^{i(\omega_1 t-k_1x)} + e^{i(\omega_2 t-k_2x)} \tag{108}\\
&= e^{i\left(\overline{\omega}t+\frac{\omega_1-\omega_2}{2}t-\overline{k}x-\frac{k_1-k_2}{2}x\right)} + e^{i\left(\overline{\omega}t-\frac{\omega_1-\omega_2}{2}t-\overline{k}x+\frac{k_1-k_2}{2}x\right)} \tag{109}\\
&= e^{i\left(\overline{\omega}t-\overline{k}x\right)}\left(e^{i\left(\frac{\omega_1-\omega_2}{2}t-\frac{k_1-k_2}{2}x\right)} + e^{i\left(-\frac{\omega_1-\omega_2}{2}t+\frac{k_1-k_2}{2}x\right)}\right) \tag{110}\\
&= e^{i\left(\overline{\omega}t-\overline{k}x\right)}2\cos\left(\frac{\omega_1-\omega_2}{2}t-\frac{k_1-k_2}{2}x\right) \tag{111}\\
&= e^{i\overline{\omega}\left(t-\frac{\overline{k}}{\overline{\omega}}x\right)}2\cos\left(\frac{\omega_1-\omega_2}{2}\left(t-\frac{k_1-k_2}{\omega_1-\omega_2}x\right)\right) \tag{112}
\end{align}$$

with $\overline{\omega} = \frac{\omega_1+\omega_2}{2}$, $\overline{k} = \frac{k_1+k_2}{2}$.

Interface Waves and Surface Waves

Dispersion

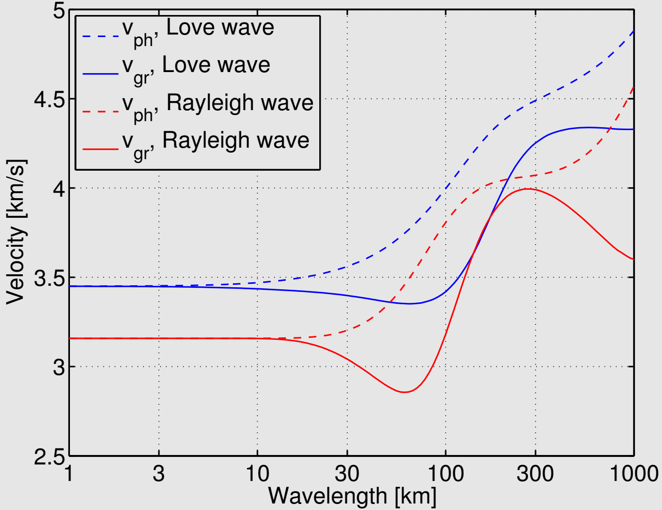
For $\omega_1 \approx \omega_2$, $k_1 \approx k_2$:

- High-frequency oscillation with an angular frequency $\overline{\omega}$ propagating with the phase slowness $s_{\mathrm{ph}} = \frac{\overline{k}}{\overline{\omega}}$
- Low-frequency oscillation of the amplitude with an angular frequency $\frac{\omega_1 - \omega_2}{2}$ propagating with the group slowness

$$s_{\mathrm{gr}} \;=\; \frac{k_1 - k_2}{\omega_1 - \omega_2} \to \frac{dk}{d\omega} \quad \text{for} \quad \omega_1 - \omega_2 \to 0 \tag{113}$$

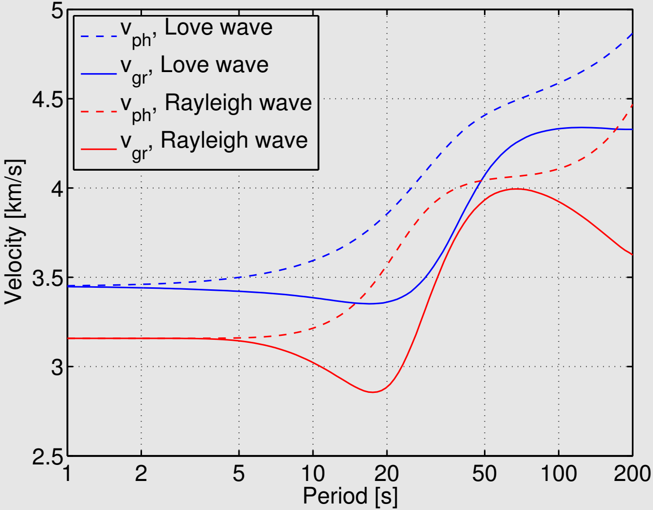
Interface Waves and Surface Waves

Velocities of Surface Waves in Typical Continental Subsurface (PEM)



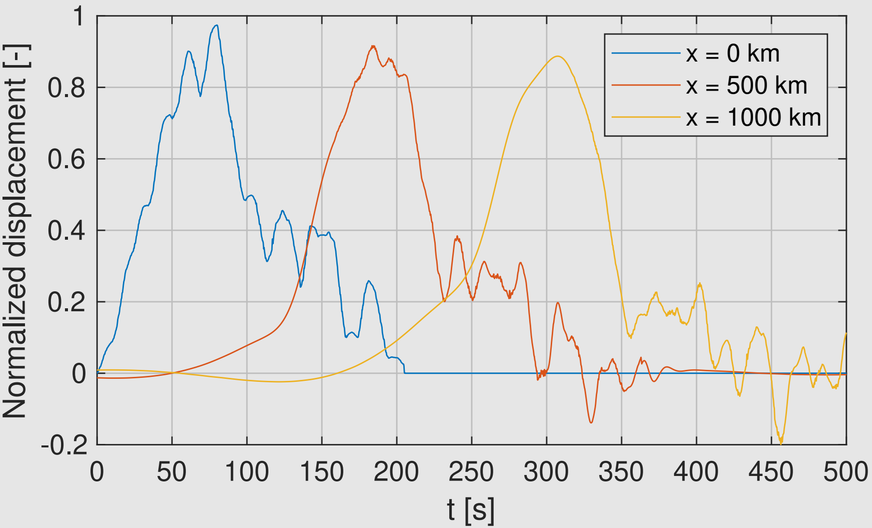
Interface Waves and Surface Waves

Velocities of Surface Waves in Typical Continental Subsurface (PEM)

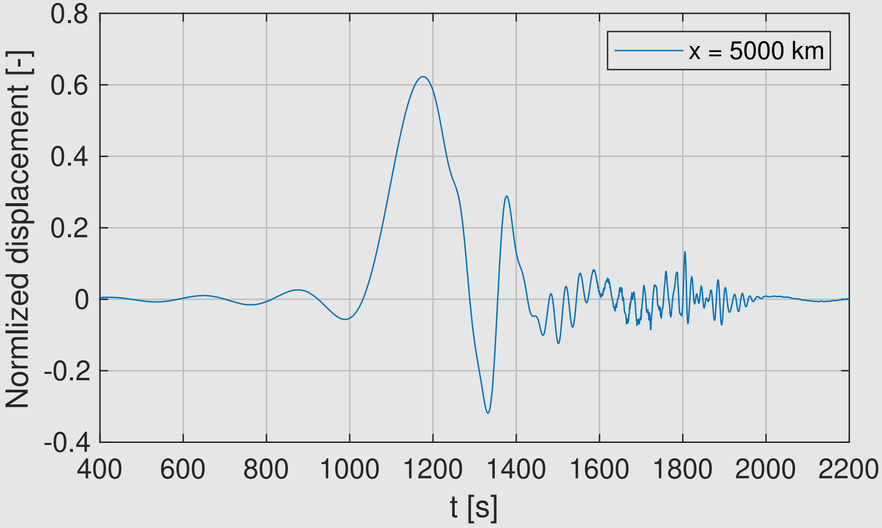


Interface Waves and Surface Waves

Dispersion of Rayleigh Waves in Typical Continental Subsurface

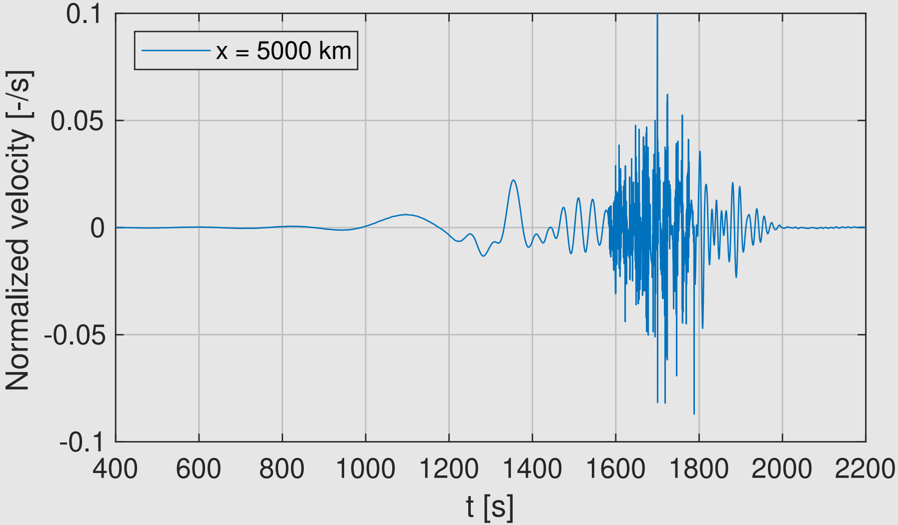


Dispersion of Rayleigh Waves in Typical Continental Subsurface

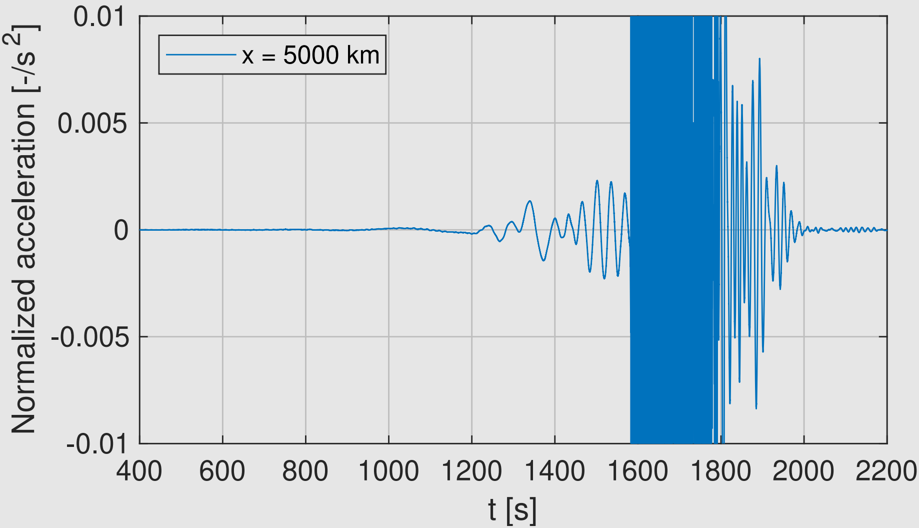


Interface Waves and Surface Waves

Dispersion of Rayleigh Waves in Typical Continental Subsurface

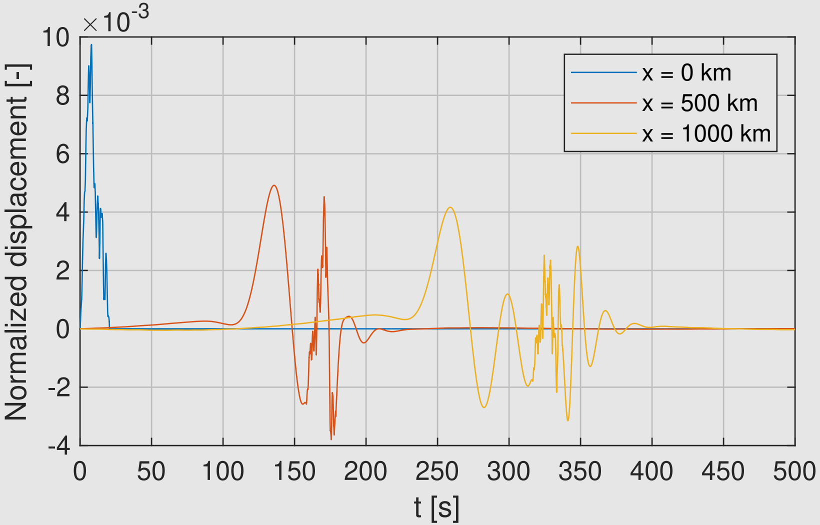


Dispersion of Rayleigh Waves in Typical Continental Subsurface



Interface Waves and Surface Waves

Dispersion of Rayleigh Waves in Typical Continental Subsurface



Interface Waves and Surface Waves

Dispersion of Rayleigh Waves in Typical Continental Subsurface