# Time-Dependent Models and Ordinary Differential Equations

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# Examples of Time-Dependent Models

## Examples From My Own Research

- Val Pola rock avalanche
- Snow avalanche hitting a pond
- Fluvial and glacial landform evolution

## Examples Considered in This Class

- Hubbert's model of oil production
- Predator-prey population dynamics
- Fluvial erosion
- Oscillations of a lamp during an earthquake
- Planetary motion and meteorite impact
- Flow of ice
- ...

# Examples of Time-Dependent Models

## Radioactive Decay

$$\frac{d}{dt}u(t) = -\lambda\, u(t)$$

where

$$\begin{aligned} t &= \text{time} \\ u(t) &= \text{amount at time } t \\ \lambda &= \text{parameter} \end{aligned}$$

### Unlimited Growth

$$\frac{d}{dt}u(t) = \lambda\, u(t)$$

Simplest model of population dynamics, where

$$u(t) = \text{population at time } t$$

# Examples of Time-Dependent Models

## Logistic Growth

$$\frac{d}{dt}u(t) = \lambda\, u(t) - \mu\, u(t)^2$$

where

$$\lambda, \mu = \text{parameters}$$

## Structure of the Previous Problems

Unknown function $u(t)$

$$+$$

Equation that defines the rate of change
$\frac{d}{dt}u(t)$ $(= u'(t) = \dot{u}(t))$

## Definition: Ordinary Differential Equation

An ordinary differential equation is an equation that involves the derivative(s) of an unknown function (and in many cases also the function itself).

## Differential Equations of First and Second Order

- A differential equation of first order involves only first-order derivatives. It can be written in the form

$$\frac{d}{dt}u(t) = F(u(t), t)$$

  and directly defines the actual rate of change in the variables.

- A second-order differential equation involves first and second-order derivatives and can be written in the form

$$\frac{d^2}{dt^2}u(t) = F\left(u(t), \frac{d}{dt}u(t), t\right).$$

# Examples of Time-Dependent Models

## Radioactive Decay Chain

$$
\begin{aligned}
\frac{d}{dt} u_1(t) &= -\lambda_1 \, u_1(t) \\
\frac{d}{dt} u_2(t) &= \lambda_1 \, u_1(t) - \lambda_2 \, u_2(t) \\
&\cdots \\
\frac{d}{dt} u_{n-1}(t) &= \lambda_{n-2} \, u_{n-2}(t) - \lambda_{n-1} \, u_{n-1}(t) \\
\frac{d}{dt} u_n(t) &= \lambda_{n-1} \, u_{n-1}(t)
\end{aligned}
$$

## Systems of Ordinary Differential Equations

Multiple unknown functions $u_1(t)$, $u_2(t)$, $\ldots$, $u_n(t)$

$$+$$

Equations that involve derivatives of these functions

## Order of a System of Differential Equations

Order of a system of differential equations

$$=$$

Order of the highest derivative that occurs

# Examples of Systems of Ordinary Differential Equations

## Chemical Reaction

$$
\begin{aligned}
\frac{d}{dt}A(t) &= -k_1\, A(t)\, B(t) + k_2\, C(t) \\
\frac{d}{dt}B(t) &= -k_1\, A(t)\, B(t) + k_2\, C(t) \\
\frac{d}{dt}C(t) &= k_1\, A(t)\, B(t) - k_2\, C(t)
\end{aligned}
$$

## SIR Model

$$
\begin{aligned}
\frac{d}{dt}S(t) &= -\beta\, S(t)\, I(t) \\
\frac{d}{dt}I(t) &= \beta\, S(t)\, I(t) - \gamma\, I(t) \\
\frac{d}{dt}R(t) &= \gamma\, I(t)
\end{aligned}
$$

## Predator-Prey Model

$$\begin{aligned}\frac{d}{dt}P(t) &= \lambda\left(1-\frac{P(t)}{c}\right)P(t) - s\,P(t)\,Q(t)\\ \frac{d}{dt}Q(t) &= \mu\left(\frac{s\,P(t)}{n}-1\right)Q(t)\end{aligned}$$

**Particle in a Gravity Field**

$$\frac{d^2}{dt^2}\vec{u}(t) = -\frac{Gm}{|\vec{u}(t)|^3}\,\vec{u}(t)$$

## Differential Equations of Second Order

General form:

$$\frac{d^2}{dt^2}u(t) = F\left(u(t), \frac{d}{dt}u(t), t\right)$$

Introduce an additional variable $v(t) = \frac{d}{dt}u(t)$.

$$\Downarrow$$

$$\begin{aligned} \frac{d}{dt}u(t) &= v(t) \\ \frac{d}{dt}v(t) &= F(u(t), v(t), t) \end{aligned}$$

## Initial Conditions

Each variable in a first-order system requires a given initial value at a time $t_0$.

## Analytical Solution of Differential Equations

Although this topic fills books and classes for engineers, only a small number of differential equations can be solved analytically.

$$\Downarrow$$

Numerical approximation required in most cases.

## The Finite-Difference Method

- An approximate solution is computed only at given times $t_1, t_2, t_3, \ldots,$ starting from the given initial state $u(t_0)$.
- Computing the solution at the time $t_{n+1}$ using the known solution at the time $t_n$ is called a (forward) time step.
- In many cases, equidistant time steps of the same length $\delta t$ are used, so that $t_1 = t_0 + \delta t$, $t_2 = t_1 + \delta t,\ t_3 = t_2 + \delta t,\ \ldots$
- Derivative $\frac{d}{dt}u(t)$ is approximated by a difference quotient.

## Right-Hand Difference Quotient

$$\frac{d}{dt}u(t) \approx \frac{u(t+\delta t) - u(t)}{\delta t}$$

## Left-Hand Difference Quotient

$$\frac{d}{dt}u(t) \approx \frac{u(t) - u(t-\delta t)}{\delta t}$$

## Central Difference Quotients

$$\frac{d}{dt}u(t) \approx \frac{u(t+\delta t) - u(t-\delta t)}{2\delta t}$$

or

$$\frac{d}{dt}u(t+\tfrac{\delta t}{2}) \approx \frac{u(t+\delta t) - u(t)}{\delta t}$$

## The Explicit Euler Scheme

Differential equation

$$\frac{d}{dt}u(t) \;=\; F(u(t), t)$$

Insert a right-hand difference quotient.

$$\Downarrow$$

$$\frac{u(t+\delta t) - u(t)}{\delta t} \;\approx\; F(u(t), t),$$

$$\Downarrow$$

$$u(t+\delta t) \;\approx\; u(t) + \delta t\, F(u(t), t)$$

## The Fully Implicit Euler Scheme

Differential equation

$$\frac{d}{dt}u(t) \;=\; F(u(t), t)$$

Insert a left-hand difference quotient.

$$\Downarrow$$

$$\frac{u(t+\delta t) - u(t)}{\delta t} \;\approx\; F(u(t+\delta t), t+\delta t)$$

$$\Downarrow$$

$$u(t+\delta t) \;\approx\; u(t) + \delta t\, F(u(t+\delta t), t+\delta t)$$

## Examples of Explicit and Implicit Discretization

Radioactive decay:

Explicit:

$$u(t+\delta t) \;\approx\; u(t) + \delta t\,(-\lambda\, u(t))$$

Fully implicit:

$$u(t+\delta t) \;\approx\; u(t) + \delta t\,(-\lambda\, u(t+\delta t))$$

$$\Downarrow$$

$$u(t+\delta t) \;\approx\; \frac{u(t)}{1+\delta t\,\lambda}$$

## Examples of Explicit and Implicit Discretization

Logistic growth:

Explicit:

$$u(t+\delta t) \;\approx\; u(t) + \delta t\,\lambda \left(1 - \frac{u(t)}{c}\right) u(t)$$

Fully implicit:

$$u(t+\delta t) \;\approx\; u(t) + \delta t\,\lambda \left(1 - \frac{u(t+\delta t)}{c}\right) u(t+\delta t)$$

$$\Downarrow$$

$$\square \; u(t+\delta t)^2 + \square \; u(t+\delta t) + \square \;=\; 0$$

## Mixed Schemes

Mixture of $t$ and $t + \delta t$ in the rate of change $F(u(t), t))$

Example: logistic growth

$$u(t + \delta t) \;\approx\; u(t) + \delta t\, \lambda \left(1 - \frac{u(t)}{c}\right) u(t + \delta t)$$

or

$$u(t + \delta t) \;\approx\; u(t) + \delta t\, \lambda \left(1 - \frac{u(t + \delta t)}{c}\right) u(t)$$

## The Crank-Nicholson Scheme

Specific mixture of explicit and fully implicit Euler scheme:

$$\frac{u(t+\delta t) - u(t)}{\delta t} \approx \frac{F(u(t), t) + F(u(t+\delta t), t+\delta t)}{2}$$

$$\Downarrow$$

$$u(t+\delta t) - \frac{\delta t}{2} F(u(t+\delta t), t+\delta t) \approx u(t) + \frac{\delta t}{2} F(u(t), t)$$

## Advantages of the Different Schemes

Explicit: simple

Fully implicit: often stable for large $\delta t$

Crank-Nicholson: high accuracy for $\delta t \to 0$;
convergence of second order, i. e., error $\propto \delta t^2$ instead of $\delta t$

## Explicit Schemes of Higher Order

An error $\propto \delta t^n$ with $n > 1$ can also be achieved by appropriate explicit schemes, e. g., by the 4th order Runge-Kutta scheme

$$u(t + \delta t) \approx u(t) + \delta t \frac{k_1 + 2k_2 + 2k_3 + k_4}{6}$$

with

$$\begin{aligned} k_1 &= F(u(t), t) \quad \text{(like explicit Euler scheme)} \\ k_2 &= F(u(t) + \tfrac{\delta t}{2} k_1, t + \tfrac{\delta t}{2}) \\ k_3 &= F(u(t) + \tfrac{\delta t}{2} k_2, t + \tfrac{\delta t}{2}) \\ k_4 &= F(u(t) + \delta t\, k_3, t + \delta t) \end{aligned}$$

# Analytical Considerations

## Motivation

Although many differential equations cannot be solved analytically, several properties of the solution can often be obtained without numerical simulations.

## Fixed Points

A fixed point is a solution which remains constant through time. The fixed points of a (system of) differential equation(s) are computed by solving

$$\frac{d}{dt}u(t) = F(u(t)) = 0.$$

# Analytical Considerations

## Stability of Fixed Points

A fixed point $u_f$ is stable if the system approaches the fixed point if it is close to it. Stable fixed points are also called attractors.

For a single differential equation of first order: $u_f$ is stable if

$$\begin{array}{ccc} F(u) & > & 0 \\ F(u) & < & 0 \end{array} \quad \text{for} \quad \begin{array}{ccc} u & < & u_f \\ u & > & u_f \end{array}$$

Alternative criterion:

$$\frac{d}{du}F(u)|_{u=u_f} \; < \; 0$$

### Stability of the Fully Implicit Euler Scheme

A time step of the fully implicity Euler scheme cannot cross a stable fixed point.

## Nondimensional Variables

Idea: If the differential equation has a characteristic time $t_c$ and / or a characteristic value $u_c$ of the solution $u(t)$, introduce nondimensional variables

$$\hat{t} = \frac{t}{t_c}$$

$$\hat{u}(\hat{t}) = \frac{u(t)}{u_c}$$

$$\Downarrow$$

$$\frac{d}{d\hat{t}}\hat{u}(\hat{t}) = \frac{t_c}{u_c}\frac{d}{dt}u(t)$$

Advantage: Each of the transforms reduces the number of model parameters by one.