Geothermics and Geothermal Energy
The Temperature in Earth's Crust

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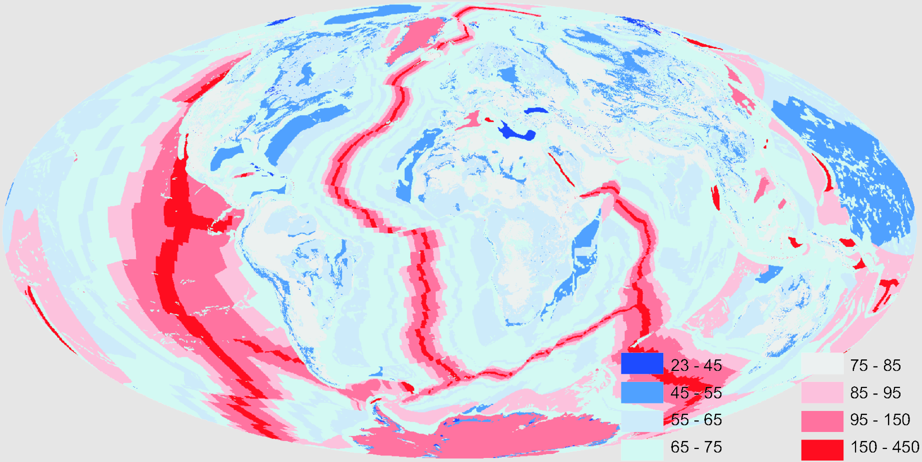
Earth’s Surface Heat Flux

Regional Variation of the Surface Heat Flux Density



Source: Davies & Davies, Earth’s surface heat flux, Solid Earth, 2010

Earth's Surface Heat Flux

Definition of Heat Flux Density

$\vec{q}$ = energy per time and cross section area $[\frac{\mathrm{W}}{\mathrm{m}^2}]$

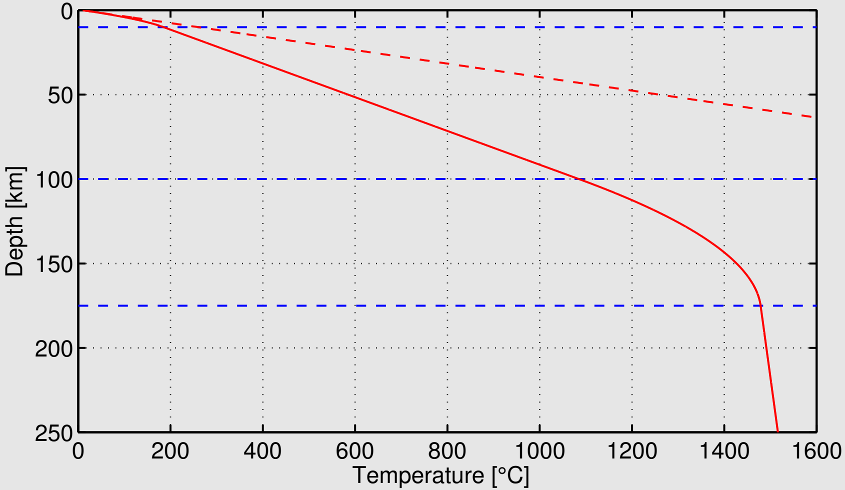
Earth's Mean Surface Heat Flux Density

Continental crust (40 % of total surface): $q = 70.9\,\frac{\mathrm{mW}}{\mathrm{m}^2}$
Oceanic crust (60 % of total surface): $q = 105.4\,\frac{\mathrm{mW}}{\mathrm{m}^2}$
Overall mean: $q = 91.6\,\frac{\mathrm{mW}}{\mathrm{m}^2}$

- q is much smaller than the solar constant $S =$ $\frac{\mathrm{W}}{\mathrm{m}^2}$.
- q describes the long-term mean energy balance ot the solid Earth.
- q reflects the ongoing cooling of Earth and from radiogenic heat production in the upper continental crust.

The Temperature in Earth's Crust

A Typical Continental Geotherm



Fourier's Law of Heat Conduction (1822)

1. Heat flow follows the direction of steepest descent of the temperature field $T(\vec{x}, t) = T(x, y, z, t)$.
2. Heat flow is proportional to the decrease of temperature per length:

$$\vec{q}(\vec{x}, t) \;=\; -\lambda \nabla T(\vec{x}, t) \;=\; -\lambda \begin{pmatrix} \frac{\partial}{\partial x} T(x, y, z, t) \\ \frac{\partial}{\partial y} T(x, y, z, t) \\ \frac{\partial}{\partial z} T(x, y, z, t) \end{pmatrix} \quad (1)$$

with

$\vec{q}(\vec{x}, t) \;=\;$ heat flux density (energy per area and time) $[\frac{\text{W}}{\text{m}^2}]$

$\lambda \;=\;$ thermal conductivity [ ]

The Thermal Conductivity

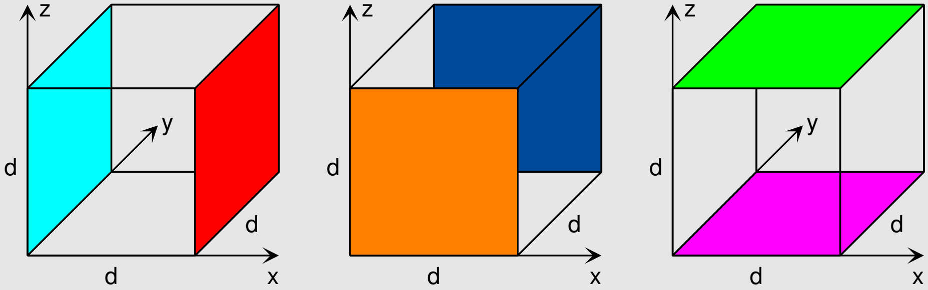
Typical Values:

Material	λ [$\frac{W}{m\,K}$]
diamond	2300
iron	80
sand	0.6
polyethylene (PE)	0.48
expanded polystyrene (EPS)	0.033
water	0.6
air	0.026

Rocks	λ [$\frac{W}{m\,K}$]
granite	2.8
basalt	2
dolomite	2.5
limestone	2.5
sandstone	2.5
clay	1.4
widely used value	

The Equation of Continuity (Energy Balance)

Energy balance of a cube without heat production:



Change in thermal energy E contained in the cube:

$$\frac{\partial E}{\partial t} = q_x d^2 - q_x d^2 + q_y d^2 - q_y d^2 + q_z d^2 - q_z d^2 \qquad (2)$$

The Equation of Continuity (Energy Balance)

Change in energy density e (thermal energy per volume):

$$\begin{aligned}\frac{\partial e}{\partial t} &= \frac{\frac{\partial E}{\partial t}}{d^3} \\ &= \frac{q_x - q_x}{d} + \frac{q_y - q_y}{d} + \frac{q_z - q_z}{d} \\ &\to -\frac{\partial q_x}{\partial x} - \frac{\partial q_y}{\partial y} - \frac{\partial q_z}{\partial z} \quad \text{for} \quad d \to 0 \\ &= -\operatorname{div}(\vec{q}) \end{aligned} \tag{3}$$

with the divergence operator

$$\operatorname{div}(\vec{q}) = \frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z} \tag{4}$$

The Heat Capacity

Volumetric heat capacity

$$c_{\mathrm{vol}} = \frac{\partial e}{\partial T} \qquad (5)$$

describes the change in thermal energy density e [ ] with T.

Unit:

Specific heat capacity

$$c = \frac{c_{\mathrm{vol}}}{\rho} = \frac{1}{\rho}\frac{\partial e}{\partial T} \qquad (6)$$

is measured per mass instead of per volume ($\rho =$ density).

Unit:

The Heat Capacity

Molar heat capacity

$$c_{\text{mol}} = Mc = \frac{M}{\rho}\frac{\partial e}{\partial T} \qquad (7)$$

is measured per mol instead of per kg (M = molar mass).

Unit:

Dulong-Petit law:

$$c_{\text{mol}} \approx 3R \qquad (8)$$

with the gas constant $R = 8.314\,\frac{\text{J}}{\text{mol K}}$ for most crystalline solids.

The Heat Capacity

Typical values at standard conditions:

Material	$c\ [\frac{\mathrm{J}}{\mathrm{kg\,K}}]$
diamond	509
iron	450
sand	550
polyethylene (PE)	1250
expanded polystyrene (EPS)	1500
water	4187
air	1005

Rocks	$c\ [\frac{\mathrm{J}}{\mathrm{kg\,K}}]$
granite	1000
basalt	850
dolomite	1000
limestone	900
sandstone	900
clay	1100
widely used value	

The Heat Conduction Equation (Energy Balance + Fourier's Law)

General version:

$$\rho c \frac{\partial T}{\partial t} = \frac{\partial e}{\partial t} = -\operatorname{div}(\vec{q}) = \operatorname{div}(\lambda \nabla T)$$
$$= \frac{\partial}{\partial x}\left(\lambda \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(\lambda \frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(\lambda \frac{\partial T}{\partial z}\right) \tag{9}$$

Simplified version for constant λ:

$$\rho c \frac{\partial T}{\partial t} = \lambda \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) = \lambda \Delta T \tag{10}$$

with the Laplace operator

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \operatorname{div}\nabla \tag{11}$$

The Thermal Diffusivity

The heat conduction equation for constant λ can be written in the form

$$\frac{\partial T}{\partial t} = \kappa \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right) = \kappa \Delta T \qquad (12)$$

with the thermal diffusivity

$$\kappa = \frac{\quad}{\quad}$$

Water: $\kappa = 1.4 \times 10^{-7} \, \frac{\text{m}^2}{\text{s}}$

Rocks: $\kappa \approx 10^{-6} \, \frac{\text{m}^2}{\text{s}} \approx 30 \, \frac{\text{m}^2}{\text{a}}$

Fundamentals – Heat Transport

Thermal Conductivity (λ)

Unit: $\frac{\mathrm{W}}{\mathrm{m\,K}}$

Meaning: Describes how well a material conducts heat.

Heat Capacity (c, c_{vol}, c_{mol})

Unit: $\frac{\mathrm{J}}{\mathrm{kg\,K}}$, $\frac{\mathrm{J}}{\mathrm{m^3K}}$, $\frac{\mathrm{J}}{\mathrm{mol\,K}}$

Meaning: Describes how much energy is needed to heat up a material.

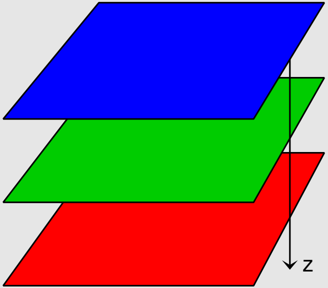
Thermal Diffusivity (κ)

Unit: $\frac{\mathrm{m^2}}{\mathrm{s}}$

Meaning: Describes how rapidly temperature propagates.

One-Dimensional Description

Most of the large-scale heat conduction problems in the lithosphere can be approximated in 1D.



$T(x, y, z, t)$ does not depend on x and y

⬇

$$\rho c \frac{\partial}{\partial t} T(z,t) = -\frac{\partial}{\partial z} q(z,t) \qquad (13)$$

$$q(z,t) = -\lambda \frac{\partial}{\partial z} T(z,t) \qquad (14)$$

The z axis is often assumed to point in downward direction.

One-Dimensional Steady-State Geotherms

$$\rho c \frac{\partial}{\partial t} T(z,t) = -\frac{\partial}{\partial z} q(z,t) = 0 \qquad (15)$$

$$\Downarrow$$

$$q(z) = -\lambda \frac{\partial}{\partial z} T(z) = -q_s = \text{const} \qquad (16)$$

with $q_s = -q(0) =$ surface heat flux density

$$\Downarrow$$

$$T(z) = T_s + \frac{q_s}{\lambda} z \qquad (17)$$

if λ is constant with $T_s = T(0) =$ surface temperature

Why are Geotherms Curved?

- Variation in λ with depth
- Advective heat transport
- Non-steady state, crust is still cooling
- Radiogenic heat production

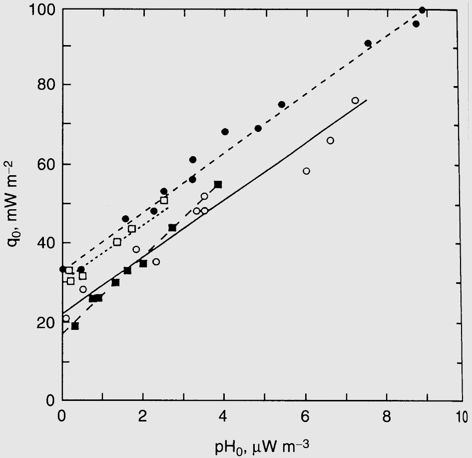
Radiogenic Heat Production

- Main contributions: decay of uranium ^{238}U, ^{235}U, thorium ^{232}Th, and potassium ^{40}K.
- Strong variation; typical heat production rates:

granite: $S \approx 2.5\,\frac{\mu\mathrm{W}}{\mathrm{m}^3}$

basalt: $S \approx 0.1\,\frac{\mu\mathrm{W}}{\mathrm{m}^3}$

Correlation of Surface Heat Flux and Heat Production



Source: Turcotte & Schubert, Geodynamics

The Heat Conduction Equation with Heat Production

3D version:

$$\rho c \frac{\partial}{\partial t} T(\vec{x}, t) = -\mathrm{div}\left(\vec{q}(\vec{x}, t)\right) + S(\vec{x}, t) \qquad (18)$$

1D version:

$$\rho c \frac{\partial}{\partial t} T(z, t) = -\frac{\partial}{\partial z} q(z, t) + S(z, t) \qquad (19)$$

Steady-State Heat Conduction with Heat Production

$$\frac{\partial}{\partial z} q(z) = S(z) \tag{20}$$

$$\Downarrow$$

$$q(z) - q(0) = \int_0^z S(\xi)\, d\xi \tag{21}$$

$$\Downarrow$$

$$q_s - q_b = -q(0) - (-q(d)) = \int_0^d S(\xi)\, d\xi \tag{22}$$

where d = thickness of the lithosphere

$q_b = -q(d)$ = basal heat flux density

One-Dimensional Steady-State Geotherms

Relationship Between Heat Flux Density and Heat Production

From the diagram:

- Variations in surface heat flux density mainly arise from variations in radiogenic heat production.
- Straight line

$$q_s = q_b + hS_s \tag{23}$$

where S_s = heat production rate at the surface.

Typical values:

$$q_b \approx 28\,\frac{\mathrm{mW}}{\mathrm{m}^2} \tag{24}$$

$$h = \frac{q_s - q_b}{S_s} \approx 10\,\mathrm{km} \tag{25}$$

How is Heat Production Distributed in the Crust?

Two simple models:

Model 1: constant heat production down to a given depth h

$$S(z) = \begin{cases} S_s & \text{for } z \leq h \\ 0 & \text{else} \end{cases} \tag{26}$$

$$\Downarrow$$

$$\begin{aligned} q(z) &= q(0) + \int_0^z S(\xi)\, d\xi \\ &= -\begin{cases} q_s - (q_s - q_b)\,\frac{z}{h} & \text{for } z \leq h \\ q_b & \text{else} \end{cases} \end{aligned} \tag{27}$$

How is Heat Production Distributed in the Crust?

Model 2: exponentially decreasing heat production rate

$$S(z) = S_s \, e^{-\frac{z}{h}} \tag{28}$$

- Both models cannot be distinguished from the surface data.
- Model 2 is consistent with surface erosion.

Analytical Solutions

A wealth of analytical solutions has been developed during the last centuries (see, e. g., Carslaw & Jaeger, Conduction of Heat in Solids).

- Restricted to specific situations
- In principle sufficient for all problems considered in this class.

Numerical Approximations

Most widely used techniques:

- Finite differences
- Finite elements

Separation of Variables

Look for solutions of the heat conduction equation

$$\frac{\partial}{\partial t} T(\vec{x}, t) \;=\; \kappa \, \Delta T(\vec{x}, t) \tag{29}$$

that can be written as a product

$$T(\vec{x}, t) \;=\; f(\vec{x})\, g(t) \tag{30}$$

$$\Downarrow$$

$$\frac{d}{dt} g(t) \;=\; \Lambda\, g(t) \qquad \text{and} \qquad \Delta f(\vec{x}) \;=\; \frac{\Lambda}{\kappa} f(\vec{x}) \tag{31}$$

with a constant Λ.

Separation of Variables

Solution for $g(t)$ with $g(0) = 1$:

$$g(t) \;=\; e^{\Lambda t} \tag{32}$$

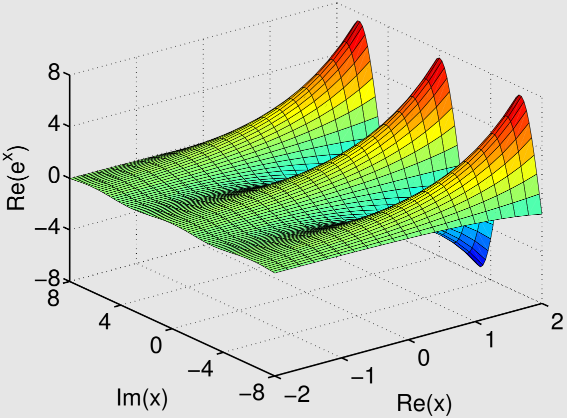
$\Lambda > 0$: Temperature increases exponentially (practically impossible).

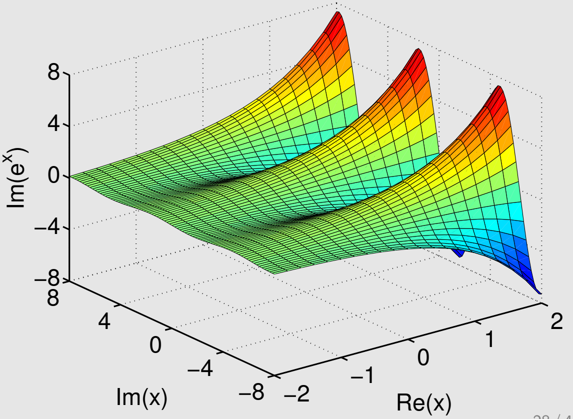
$\Lambda < 0$: Temperature decreases exponentially.

Most general solution: complex number Λ; combination of increase / decrease and harmonic oscillation

The Complex Exponential Function

With $e^{i\phi} = \cos\phi + i\sin\phi$, the complex exponential function combines the real exponential function with the sine and cosine functions.





Harmonic Solution

Assume a harmonic oscillation of the surface temperature ($z = 0$):

$$T(0, t) = \cos(\omega t) \quad (33)$$

with

$$\omega = \frac{2\pi}{\tau} = \text{angular frequency}$$
$$\tau = \text{period (e. g., 1 year)}$$

Separation approach with $\Lambda = i\omega$:

$$T(z, t) = f(z)\, e^{i\omega t} \quad (34)$$

⬇

$$\frac{\partial^2}{\partial z^2} f(z) = \frac{\Lambda}{\kappa} f(z) = \frac{i\omega}{\kappa} f(z) \quad (35)$$

Harmonic Solution

Solution with $f(0) = 1$:

$$f(z) \;=\; e^{\pm\sqrt{\frac{i\omega}{\kappa}}z} \;=\; e^{\pm(1+i)\sqrt{\frac{\omega}{2\kappa}}z} \qquad (36)$$

↓

$$\begin{aligned} T(z,t) &= e^{i\omega t}\, e^{\pm(1+i)\sqrt{\frac{\omega}{2\kappa}}z} \\ &= e^{i\left(\omega t \pm \sqrt{\frac{\omega}{2\kappa}}z\right)}\, e^{\pm\sqrt{\frac{\omega}{2\kappa}}z} \qquad (37) \end{aligned}$$

Only the version with the minus sign is physically reasonable.

Harmonic Solution

$$d = \sqrt{\frac{2\kappa}{\omega}} = \sqrt{\frac{\kappa T}{\pi}} \tag{38}$$

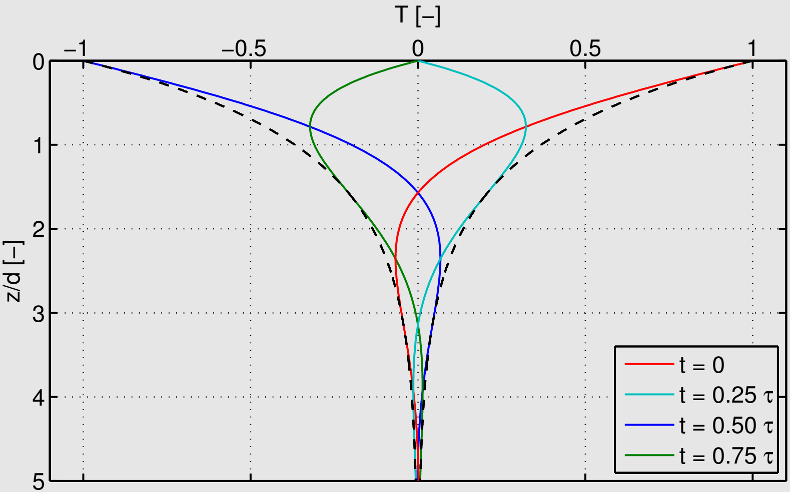
defines the depth of penetration:

$$T(z,t) = e^{i\left(\omega t - \frac{z}{d}\right)} e^{-\frac{z}{d}} \tag{39}$$

Solution with cosine instead of the complex exponential function:

$$T(z,t) = \cos\left(\omega t - \frac{z}{d}\right) e^{-\frac{z}{d}} \tag{40}$$

Harmonic Solution



Harmonic Solution

Amplitude: $e^{-\frac{z}{d}}$ $\Rightarrow$ d is the depth where the temperature oscillation has decreased to   of the oscillation at the surface.

Phase shift: Oscillation is opposite to the oscillation at the surface for $z = \pi d$.

The Diel Variation

Same equations as for the seasonal variation, but the depth of penetration d is almost   times lower.

Superposition of Solutions

The heat conduction equation

$$\rho c \frac{\partial}{\partial t} T(\vec{x}, t) = \operatorname{div}\left(\lambda \nabla T(\vec{x}, t)\right) + S(\vec{x}, t) \tag{41}$$

is linear.

$$\Downarrow$$

Solutions can be superposed.

Superposition of Solutions

- $T(\vec{x}, t)$ is a solution for $S(\vec{x}, t)$

$$\Downarrow$$

$\alpha T(\vec{x}, t)$ is a solution for

- If $T_1(\vec{x}, t)$ is a solution for $S_1(\vec{x}, t)$ and $T_2(\vec{x}, t)$ is a solution for $S_2(\vec{x}, t)$

$$\Downarrow$$

$T_1(\vec{x}, t) + T_2(\vec{x}, t)$ is a solution for

Typical Application

Assume that

- $T_m(\vec{x})$ is a solution of the steady-state equation with heat production

$$\operatorname{div}\left(\lambda \nabla T_m(\vec{x})\right) + S(\vec{x}) = 0 \tag{42}$$

- $T_t(\vec{x}, t)$ is a solution of the time-dependent equation without heat production

$$\rho c \frac{\partial}{\partial t} T_t(\vec{x}, t) = \operatorname{div}\left(\lambda \nabla T_t(\vec{x}, t)\right) \tag{43}$$

$$\Downarrow$$

$T(\vec{x}, t) = T_m(\vec{x}) + T_t(\vec{x}, t)$ is a solution of the full equation

$$\rho c \frac{\partial}{\partial t} T(\vec{x}, t) = \operatorname{div}\left(\lambda \nabla T(\vec{x}, t)\right) + S(\vec{x}) \tag{44}$$

Seasonal Variation of the Subsurface Temperature

Superposition of Seasonal and Diel Oscillation

$$T(z,t) = T_m(z) + T_y(z,t) + T_d(z,t) \quad (45)$$

with

$$\begin{aligned}
T_m(z) &= \text{steady-state geotherm} \\
T_y(z,t) &= a_y \cos\left(\omega_y(t-t_y) - \frac{z}{d_y}\right) e^{-\frac{z}{d_y}} = \text{seasonal variation} \quad (46) \\
T_d(z,t) &= a_d \cos\left(\omega_d(t-t_d) - \frac{z}{d_d}\right) e^{-\frac{z}{d_d}} = \text{diel variation} \quad (47) \\
a_y, a_d &= \text{amplitudes} \\
\omega_y, \omega_d &= \text{angular frequencies} \\
t_y, t_d &= \text{time lag of maximum temperature vs. } t = 0 \\
d_y, d_d &= \text{depths of penetration}
\end{aligned}$$

Solution for a Sudden Change in Surface Temperature

Consider 1D heat conduction in the domain $z \geq 0$ with

- initial temperature $T(z, 0) = T_0$ and
- surface temperature switches to $T(0, t) = T_s$ at $t = 0$.

Relevant for many technical applications and, e. g., for the cooling of oceanic lithosphere.

Variables z and t cannot be separated here, but a scaling relation between z and t can be used:

$$L(t) = \sqrt{\kappa t} \tag{48}$$

can be seen as a time-dependent length scale, called length scale of heat conduction.

Solution for a Sudden Change in Surface Temperature

Assume that the temperature only depends on the nondimensional variable

$$u = \frac{z}{2L(t)} = \frac{z}{2\sqrt{\kappa t}} \tag{49}$$

instead of z and t individually.

Interpretation: Shape of the temperature profile remains constant, while only the spatial scale changes.

Solution for a Sudden Change in Surface Temperature

Solution:

$$T(z,t) = T_s + (T_0 - T_s)\,\mathrm{erf}(u) = T_0 + (T_s - T_0)\,\mathrm{erfc}(u) \tag{50}$$

with the Gaussian error function

$$\mathrm{erf}(u) = \frac{2}{\sqrt{\pi}} \int\limits_0^u e^{-x^2} dx \tag{51}$$

and the complementary error function

$$\mathrm{erfc}(u) = \frac{2}{\sqrt{\pi}} \int\limits_u^\infty e^{-x^2} dx = 1 - \mathrm{erf}(u) \tag{52}$$

Solution for a Sudden Change in Surface Temperature